

Image copyright Monkey Business Images 2010.
Used under license from Shutterstock.com



FUNDAMENTALS

1.1 Real Numbers

1.2 Exponents and Radicals

1.3 Algebraic Expressions

1.4 Rational Expressions

1.5 Equations

1.6 Modeling with Equations

1.7 Inequalities

1.8 Coordinate Geometry

1.9 Graphing Calculators; Solving Equations and Inequalities Graphically

1.10 Lines

1.11 Making Models Using Variation

FOCUS ON MODELING

Fitting Lines to Data

In this first chapter we review the real numbers, equations, and the coordinate plane. You are probably already familiar with these concepts, but it is helpful to get a fresh look at how these ideas work together to solve problems and model (or describe) real-world situations.

Let's see how all these ideas are used in a real-life situation: Suppose you get paid \$9 an hour at your part-time job. We can *model* your pay y for working x hours by the *equation* $y = 9x$. To find out how many hours you need to work to get paid 200 dollars, we solve the equation $200 = 9x$. Graphing the equation $y = 9x$ in a *coordinate plane* helps us “see” how pay increases with hours worked.

1.1 REAL NUMBERS

Properties of Real Numbers ► Addition and Subtraction ► Multiplication and Division ► The Real Line ► Sets and Intervals ► Absolute Value and Distance

The different types of real numbers were invented to meet specific needs. For example, natural numbers are needed for counting, negative numbers for describing debt or below-zero temperatures, rational numbers for concepts like “half a gallon of milk,” and irrational numbers for measuring certain distances, like the diagonal of a square.

Let’s review the types of numbers that make up the real number system. We start with the **natural numbers**:

$$1, 2, 3, 4, \dots$$

The **integers** consist of the natural numbers together with their negatives and 0:

$$\dots, -3, -2, -1, 0, 1, 2, 3, 4, \dots$$

We construct the **rational numbers** by taking ratios of integers. Thus, any rational number r can be expressed as

$$r = \frac{m}{n}$$

where m and n are integers and $n \neq 0$. Examples are

$$\frac{1}{2}, \quad -\frac{3}{7}, \quad 46 = \frac{46}{1}, \quad 0.17 = \frac{17}{100}$$

(Recall that division by 0 is always ruled out, so expressions like $\frac{3}{0}$ and $\frac{0}{0}$ are undefined.) There are also real numbers, such as $\sqrt{2}$, that cannot be expressed as a ratio of integers and are therefore called **irrational numbers**. It can be shown, with varying degrees of difficulty, that these numbers are also irrational:

$$\sqrt{3}, \quad \sqrt{5}, \quad \sqrt[3]{2}, \quad \pi, \quad \frac{3}{\pi^2}$$

The set of all real numbers is usually denoted by the symbol $\mathbb{R}$. When we use the word *number* without qualification, we will mean “real number.” Figure 1 is a diagram of the types of real numbers that we work with in this book.

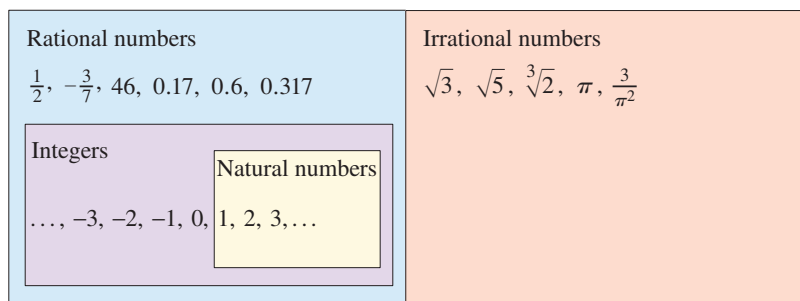


FIGURE 1 The real number system

A repeating decimal such as

$$x = 3.5474747 \dots$$

is a rational number. To convert it to a ratio of two integers, we write

$$\begin{array}{r} 1000x = 3547.47474747 \dots \\ 10x = 35.47474747 \dots \\ \hline 990x = 3512.0 \end{array}$$

Thus, $x = \frac{3512}{990}$. (The idea is to multiply x by appropriate powers of 10 and then subtract to eliminate the repeating part.)

Every real number has a decimal representation. If the number is rational, then its corresponding decimal is repeating. For example,

$$\begin{array}{ll} \frac{1}{2} = 0.5000 \dots = 0.5\overline{0} & \frac{2}{3} = 0.66666 \dots = 0.\overline{6} \\ \frac{157}{495} = 0.3171717 \dots = 0.31\overline{7} & \frac{9}{7} = 1.285714285714 \dots = 1.\overline{285714} \end{array}$$

(The bar indicates that the sequence of digits repeats forever.) If the number is irrational, the decimal representation is nonrepeating:

$$\sqrt{2} = 1.414213562373095 \dots \quad \pi = 3.141592653589793 \dots$$

If we stop the decimal expansion of any number at a certain place, we get an approximation to the number. For instance, we can write

$$\pi \approx 3.14159265$$

where the symbol $\approx$ is read “is approximately equal to.” The more decimal places we retain, the better our approximation.

▼ Properties of Real Numbers

We all know that $2 + 3 = 3 + 2$, and $5 + 7 = 7 + 5$, and $513 + 87 = 87 + 513$, and so on. In algebra, we express all these (infinitely many) facts by writing

$$a + b = b + a$$

where a and b stand for any two numbers. In other words, “ $a + b = b + a$ ” is a concise way of saying that “when we add two numbers, the order of addition doesn’t matter.” This fact is called the *Commutative Property* for addition. From our experience with numbers we know that the properties in the following box are also valid.

PROPERTIES OF REAL NUMBERS		
Property	Example	Description
Commutative Properties		
$a + b = b + a$	$7 + 3 = 3 + 7$	When we add two numbers, order doesn’t matter.
$ab = ba$	$3 \cdot 5 = 5 \cdot 3$	When we multiply two numbers, order doesn’t matter.
Associative Properties		
$(a + b) + c = a + (b + c)$	$(2 + 4) + 7 = 2 + (4 + 7)$	When we add three numbers, it doesn’t matter which two we add first.
$(ab)c = a(bc)$	$(3 \cdot 7) \cdot 5 = 3 \cdot (7 \cdot 5)$	When we multiply three numbers, it doesn’t matter which two we multiply first.
Distributive Property		
$a(b + c) = ab + ac$	$2 \cdot (3 + 5) = 2 \cdot 3 + 2 \cdot 5$	When we multiply a number by a sum of two numbers, we get the same result as we would get if we multiply the number by each of the terms and then add the results.
$(b + c)a = ab + ac$	$(3 + 5) \cdot 2 = 2 \cdot 3 + 2 \cdot 5$	

The Distributive Property applies whenever we multiply a number by a sum. Figure 2 explains why this property works for the case in which all the numbers are positive integers, but the property is true for any real numbers a , b , and c .

The Distributive Property is crucial because it describes the way addition and multiplication interact with each other.

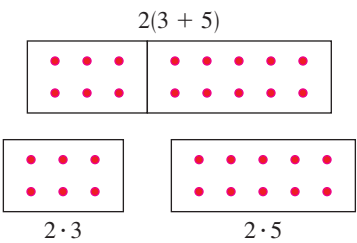


FIGURE 2 The Distributive Property

EXAMPLE 1 | Using the Distributive Property

$$\begin{aligned}
 \text{(a)} \quad & 2(x + 3) = 2 \cdot x + 2 \cdot 3 && \text{Distributive Property} \\
 & = 2x + 6 && \text{Simplify} \\
 \text{(b)} \quad & (a + b)(x + y) = (a + b)x + (a + b)y && \text{Distributive Property} \\
 & = (ax + bx) + (ay + by) && \text{Distributive Property} \\
 & = ax + bx + ay + by && \text{Associative Property of Addition}
 \end{aligned}$$

In the last step we removed the parentheses because, according to the Associative Property, the order of addition doesn't matter.

 NOW TRY EXERCISE 11

 Don't assume that $-a$ is a negative number. Whether $-a$ is negative or positive depends on the value of a . For example, if $a = 5$, then $-a = -5$, a negative number, but if $a = -5$, then $-a = -(-5) = 5$ (Property 2), a positive number.

▼ Addition and Subtraction

The number 0 is special for addition; it is called the **additive identity** because $a + 0 = a$ for any real number a . Every real number a has a **negative**, $-a$, that satisfies $a + (-a) = 0$. **Subtraction** is the operation that undoes addition; to subtract a number from another, we simply add the negative of that number. By definition

$$a - b = a + (-b)$$

To combine real numbers involving negatives, we use the following properties.

PROPERTIES OF NEGATIVES

Property	Example
1. $(-1)a = -a$	$(-1)5 = -5$
2. $-(-a) = a$	$-(-5) = 5$
3. $(-a)b = a(-b) = -(ab)$	$(-5)7 = 5(-7) = -(5 \cdot 7)$
4. $(-a)(-b) = ab$	$(-4)(-3) = 4 \cdot 3$
5. $-(a + b) = -a - b$	$-(3 + 5) = -3 - 5$
6. $-(a - b) = b - a$	$-(5 - 8) = 8 - 5$

Property 6 states the intuitive fact that $a - b$ and $b - a$ are negatives of each other. Property 5 is often used with more than two terms:

$$-(a + b + c) = -a - b - c$$

EXAMPLE 2 | Using Properties of Negatives

Let x , y , and z be real numbers.

$$\begin{aligned}
 \text{(a)} \quad & -(x + 2) = -x - 2 && \text{Property 5: } -(a + b) = -a - b \\
 \text{(b)} \quad & -(x + y - z) = -x - y - (-z) && \text{Property 5: } -(a + b) = -a - b \\
 & = -x - y + z && \text{Property 2: } -(-a) = a
 \end{aligned}$$

 NOW TRY EXERCISE 23

▼ Multiplication and Division

The number 1 is special for multiplication; it is called the **multiplicative identity** because $a \cdot 1 = a$ for any real number a . Every nonzero real number a has an **inverse**, $1/a$, that satisfies $a \cdot (1/a) = 1$. **Division** is the operation that undoes multiplication; to divide by a number, we multiply by the inverse of that number. If $b \neq 0$, then, by definition,

$$a \div b = a \cdot \frac{1}{b}$$

We write $a \cdot (1/b)$ as simply a/b . We refer to a/b as the **quotient** of a and b or as the **fraction** a over b ; a is the **numerator** and b is the **denominator** (or **divisor**). To combine real numbers using the operation of division, we use the following properties.

PROPERTIES OF FRACTIONS

Property	Example	Description
1. $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$	$\frac{2}{3} \cdot \frac{5}{7} = \frac{2 \cdot 5}{3 \cdot 7} = \frac{10}{21}$	When multiplying fractions , multiply numerators and denominators.
2. $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c}$	$\frac{2}{3} \div \frac{5}{7} = \frac{2}{3} \cdot \frac{7}{5} = \frac{14}{15}$	When dividing fractions , invert the divisor and multiply.
3. $\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c}$	$\frac{2}{5} + \frac{7}{5} = \frac{2+7}{5} = \frac{9}{5}$	When adding fractions with the same denominator , add the numerators.
4. $\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$	$\frac{2}{5} + \frac{3}{7} = \frac{2 \cdot 7 + 3 \cdot 5}{35} = \frac{29}{35}$	When adding fractions with different denominators , find a common denominator. Then add the numerators.
5. $\frac{ac}{bc} = \frac{a}{b}$	$\frac{2 \cdot 5}{3 \cdot 5} = \frac{2}{3}$	Cancel numbers that are common factors in numerator and denominator.
6. If $\frac{a}{b} = \frac{c}{d}$, then $ad = bc$	$\frac{2}{3} = \frac{6}{9}$, so $2 \cdot 9 = 3 \cdot 6$	Cross-multiply .

When adding fractions with different denominators, we don't usually use Property 4. Instead we rewrite the fractions so that they have the smallest possible common denominator (often smaller than the product of the denominators), and then we use Property 3. This denominator is the **Least Common Denominator (LCD)** described in the next example.

EXAMPLE 3 | Using the LCD to Add Fractions

Evaluate: $\frac{5}{36} + \frac{7}{120}$

SOLUTION Factoring each denominator into prime factors gives

$$36 = 2^2 \cdot 3^2 \quad \text{and} \quad 120 = 2^3 \cdot 3 \cdot 5$$

We find the least common denominator (LCD) by forming the product of all the factors that occur in these factorizations, using the highest power of each factor.

Thus the LCD is $2^3 \cdot 3^2 \cdot 5 = 360$. So

$$\begin{aligned}\frac{5}{36} + \frac{7}{120} &= \frac{5 \cdot 10}{36 \cdot 10} + \frac{7 \cdot 3}{120 \cdot 3} && \text{Use common denominator} \\ &= \frac{50}{360} + \frac{21}{360} = \frac{71}{360} && \text{Property 3: Adding fractions with the same denominator}\end{aligned}$$

 NOW TRY EXERCISE 25

▼ The Real Line

The real numbers can be represented by points on a line, as shown in Figure 3. The positive direction (toward the right) is indicated by an arrow. We choose an arbitrary reference point O , called the **origin**, which corresponds to the real number 0. Given any convenient unit of measurement, each positive number x is represented by the point on the line a distance of x units to the right of the origin, and each negative number $-x$ is represented by the point x units to the left of the origin. The number associated with the point P is called the **coordinate of P** , and the line is then called a **coordinate line**, or a **real number line**, or simply a **real line**. Often we identify the point with its coordinate and think of a number as being a point on the real line.

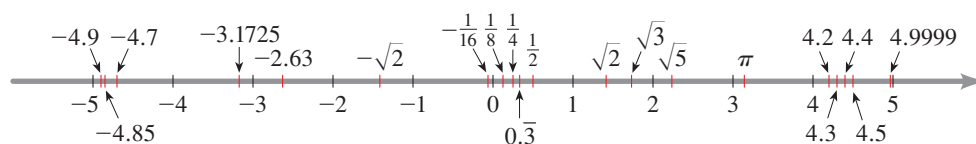


FIGURE 3 The real line

The real numbers are *ordered*. We say that **a is less than b** and write $a < b$ if $b - a$ is a positive number. Geometrically, this means that a lies to the left of b on the number line. Equivalently, we can say that **b is greater than a** and write $b > a$. The symbol $a \leq b$ (or $b \geq a$) means that either $a < b$ or $a = b$ and is read “ a is less than or equal to b .” For instance, the following are true inequalities (see Figure 4):

$$7 < 7.4 < 7.5 \qquad -\pi < -3 \qquad \sqrt{2} < 2 \qquad 2 \leq 2$$

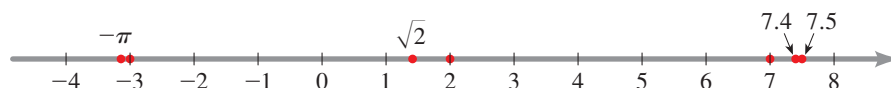


FIGURE 4

▼ Sets and Intervals

A **set** is a collection of objects, and these objects are called the **elements** of the set. If S is a set, the notation $a \in S$ means that a is an element of S , and $b \notin S$ means that b is not an element of S . For example, if Z represents the set of integers, then $-3 \in Z$ but $\pi \notin Z$.

Some sets can be described by listing their elements within braces. For instance, the set A that consists of all positive integers less than 7 can be written as

$$A = \{1, 2, 3, 4, 5, 6\}$$

We could also write A in **set-builder notation** as

$$A = \{x \mid x \text{ is an integer and } 0 < x < 7\}$$

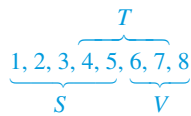
which is read “ A is the set of all x such that x is an integer and $0 < x < 7$.”

If S and T are sets, then their **union** $S \cup T$ is the set that consists of all elements that are in S or T (or in both). The **intersection** of S and T is the set $S \cap T$ consisting of all elements that are in both S and T . In other words, $S \cap T$ is the common part of S and T . The **empty set**, denoted by $\emptyset$, is the set that contains no element.

EXAMPLE 4 | Union and Intersection of Sets

If $S = \{1, 2, 3, 4, 5\}$, $T = \{4, 5, 6, 7\}$, and $V = \{6, 7, 8\}$, find the sets $S \cup T$, $S \cap T$, and $S \cap V$.

SOLUTION



$$\begin{aligned} S \cup T &= \{1, 2, 3, 4, 5, 6, 7\} && \text{All elements in } S \text{ or } T \\ S \cap T &= \{4, 5\} && \text{Elements common to both } S \text{ and } T \\ S \cap V &= \emptyset && S \text{ and } V \text{ have no element in common} \end{aligned}$$

NOW TRY EXERCISE 39



FIGURE 5 The open interval (a, b)



FIGURE 6 The closed interval $[a, b]$

Certain sets of real numbers, called **intervals**, occur frequently in calculus and correspond geometrically to line segments. If $a < b$, then the **open interval** from a to b consists of all numbers between a and b and is denoted (a, b) . The **closed interval** from a to b includes the endpoints and is denoted $[a, b]$. Using set-builder notation, we can write

$$(a, b) = \{x \mid a < x < b\} \quad [a, b] = \{x \mid a \leq x \leq b\}$$

Note that parentheses $()$ in the interval notation and open circles on the graph in Figure 5 indicate that endpoints are *excluded* from the interval, whereas square brackets $[]$ and solid circles in Figure 6 indicate that the endpoints are *included*. Intervals may also include one endpoint but not the other, or they may extend infinitely far in one direction or both. The following table lists the possible types of intervals.

Notation	Set description	Graph
(a, b)	$\{x \mid a < x < b\}$	
$[a, b]$	$\{x \mid a \leq x \leq b\}$	
$[a, b)$	$\{x \mid a \leq x < b\}$	
$(a, b]$	$\{x \mid a < x \leq b\}$	
(a, ∞)	$\{x \mid a < x\}$	
$[a, \infty)$	$\{x \mid a \leq x\}$	
$(-\infty, b)$	$\{x \mid x < b\}$	
$(-\infty, b]$	$\{x \mid x \leq b\}$	
$(-\infty, \infty)$	$\mathbb{R}$ (set of all real numbers)	

The symbol ∞ (“infinity”) does not stand for a number. The notation (a, ∞) , for instance, simply indicates that the interval has no endpoint on the right but extends infinitely far in the positive direction.

EXAMPLE 5 | Graphing Intervals

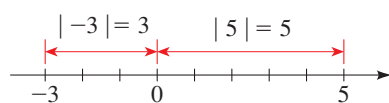
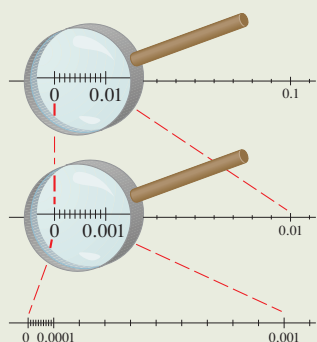
Express each interval in terms of inequalities, and then graph the interval.

- (a) $[-1, 2) = \{x \mid -1 \leq x < 2\}$
- (b) $[1.5, 4] = \{x \mid 1.5 \leq x \leq 4\}$
- (c) $(-3, \infty) = \{x \mid -3 < x\}$

NOW TRY EXERCISE 45

No Smallest or Largest Number in an Open Interval

Any interval contains infinitely many numbers—every point on the graph of an interval corresponds to a real number. In the closed interval $[0, 1]$, the smallest number is 0 and the largest is 1, but the open interval $(0, 1)$ contains no smallest or largest number. To see this, note that 0.01 is close to zero, but 0.001 is closer, 0.0001 is closer yet, and so on. We can always find a number in the interval $(0, 1)$ closer to zero than any given number. Since 0 itself is not in the interval, the interval contains no smallest number. Similarly, 0.99 is close to 1, but 0.999 is closer, 0.9999 closer yet, and so on. Since 1 itself is not in the interval, the interval has no largest number.

**FIGURE 9****EXAMPLE 6** | Finding Unions and Intersections of Intervals

Graph each set.

- (a) $(1, 3) \cap [2, 7]$ (b) $(1, 3) \cup [2, 7]$

SOLUTION

- (a) The intersection of two intervals consists of the numbers that are in both intervals. Therefore

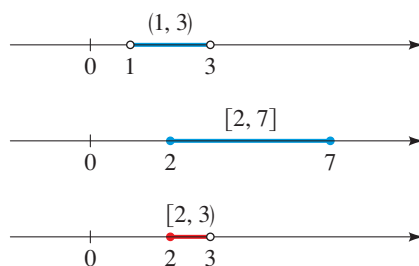
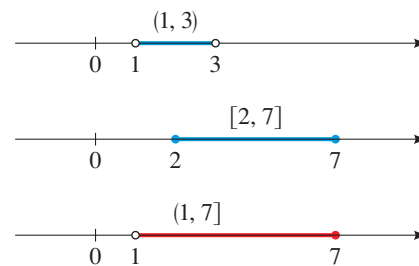
$$\begin{aligned}(1, 3) \cap [2, 7] &= \{x \mid 1 < x < 3 \text{ and } 2 \leq x \leq 7\} \\ &= \{x \mid 2 \leq x < 3\} = [2, 3)\end{aligned}$$

This set is illustrated in Figure 7.

- (b) The union of two intervals consists of the numbers that are in either one interval or the other (or both). Therefore

$$\begin{aligned}(1, 3) \cup [2, 7] &= \{x \mid 1 < x < 3 \text{ or } 2 \leq x \leq 7\} \\ &= \{x \mid 1 < x \leq 7\} = (1, 7]\end{aligned}$$

This set is illustrated in Figure 8.

**FIGURE 7** $(1, 3) \cap [2, 7] = [2, 3)$ **FIGURE 8** $(1, 3) \cup [2, 7] = (1, 7]$ **NOW TRY EXERCISE 59****▼ Absolute Value and Distance**

The **absolute value** of a number a , denoted by $|a|$, is the distance from a to 0 on the real number line (see Figure 9). Distance is always positive or zero, so we have $|a| \geq 0$ for every number a . Remembering that $-a$ is positive when a is negative, we have the following definition.

DEFINITION OF ABSOLUTE VALUEIf a is a real number, then the **absolute value** of a is

$$|a| = \begin{cases} a & \text{if } a \geq 0 \\ -a & \text{if } a < 0 \end{cases}$$

EXAMPLE 7 | Evaluating Absolute Values of Numbers

- (a) $|3| = 3$
 (b) $|-3| = -(-3) = 3$
 (c) $|0| = 0$
 (d) $|3 - \pi| = -(3 - \pi) = \pi - 3$ (since $3 < \pi \Rightarrow 3 - \pi < 0$)

NOW TRY EXERCISE 65

When working with absolute values, we use the following properties.

PROPERTIES OF ABSOLUTE VALUE

Property	Example	Description
1. $ a \geq 0$	$ -3 = 3 \geq 0$	The absolute value of a number is always positive or zero.
2. $ a = -a $	$ 5 = -5 $	A number and its negative have the same absolute value.
3. $ ab = a b $	$ -2 \cdot 5 = -2 5 $	The absolute value of a product is the product of the absolute values.
4. $\left \frac{a}{b}\right = \frac{ a }{ b }$	$\left \frac{12}{-3}\right = \frac{ 12 }{ -3 }$	The absolute value of a quotient is the quotient of the absolute values.

What is the distance on the real line between the numbers -2 and 11 ? From Figure 10 we see that the distance is 13 . We arrive at this by finding either $|11 - (-2)| = 13$ or $|(-2) - 11| = 13$. From this observation we make the following definition (see Figure 11).

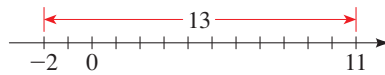


FIGURE 10

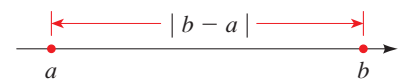


FIGURE 11 Length of a line segment is $|b - a|$

DISTANCE BETWEEN POINTS ON THE REAL LINE

If a and b are real numbers, then the **distance** between the points a and b on the real line is

$$d(a, b) = |b - a|$$

From Property 6 of negatives it follows that

$$|b - a| = |a - b|$$

This confirms that, as we would expect, the distance from a to b is the same as the distance from b to a .

EXAMPLE 8 | Distance Between Points on the Real Line

The distance between the numbers -8 and 2 is

$$d(a, b) = |-8 - 2| = |-10| = 10$$

We can check this calculation geometrically, as shown in Figure 12.

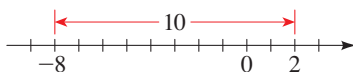


FIGURE 12

 **NOW TRY EXERCISE 73**

1.1 EXERCISES

CONCEPTS

- Give an example of each of the following:
 - A natural number
 - An integer that is not a natural number
 - A rational number that is not an integer
 - An irrational number
- Complete each statement and name the property of real numbers you have used.
 - $ab =$ _____; _____ Property
 - $a + (b + c) =$ _____; _____ Property
 - $a(b + c) =$ _____; _____ Property
- The set of numbers between but not including 2 and 7 can be written as follows:

_____ in set-builder notation and
_____ in interval notation.
- The symbol $|x|$ stands for the _____ of the number x . If x is not 0, then the sign of $|x|$ is always _____.

SKILLS

- 5–6 ■ List the elements of the given set that are

- natural numbers
- integers
- rational numbers
- irrational numbers

- $\{0, -10, 50, \frac{22}{7}, 0.538, \sqrt{7}, 1.2\bar{3}, -\frac{1}{3}, \sqrt[3]{2}\}$
- $\{1.001, 0.333 \dots, -\pi, -11, 11, \frac{13}{15}, \sqrt{16}, 3.14, \frac{15}{3}\}$

- 7–14 ■ State the property of real numbers being used.

- $7 + 10 = 10 + 7$
- $2(3 + 5) = (3 + 5)2$
- $(x + 2y) + 3z = x + (2y + 3z)$
- $2(A + B) = 2A + 2B$
- $(5x + 1)3 = 15x + 3$
- $(x + a)(x + b) = (x + a)x + (x + a)b$
- $2x(3 + y) = (3 + y)2x$
- $7(a + b + c) = 7(a + b) + 7c$

- 15–18 ■ Rewrite the expression using the given property of real numbers.

- Commutative Property of addition, $x + 3 =$ _____
- Associative Property of multiplication, $7(3x) =$ _____
- Distributive Property, $4(A + B) =$ _____
- Distributive Property, $5x + 5y =$ _____

- 19–24 ■ Use properties of real numbers to write the expression without parentheses.

- $3(x + y)$
- $4(2m)$
- $-\frac{5}{2}(2x - 4y)$
- $(a - b)8$
- $\frac{4}{3}(-6y)$
- $(3a)(b + c - 2d)$

- 25–30 ■ Perform the indicated operations.

- $\frac{3}{10} + \frac{4}{15}$
- $\frac{2}{3} - \frac{3}{5}$
- $\frac{2}{3}(6 - \frac{3}{2})$
- $(3 + \frac{1}{4})(1 - \frac{4}{5})$
- $\frac{2}{\frac{2}{3}} - \frac{\frac{2}{3}}{2}$
- $\frac{2 - \frac{3}{4}}{\frac{1}{2} - \frac{1}{3}}$
- $\frac{1}{4} + \frac{1}{5}$
- $1 + \frac{5}{8} - \frac{1}{6}$
- $0.25(\frac{8}{9} + \frac{1}{2})$
- $(\frac{1}{2} - \frac{1}{3})(\frac{1}{2} + \frac{1}{3})$
- $\frac{\frac{1}{12}}{\frac{1}{8} - \frac{1}{9}}$
- $\frac{\frac{2}{5} + \frac{1}{2}}{\frac{1}{10} + \frac{3}{15}}$

- 31–32 ■ Place the correct symbol ($<$, $>$, or $=$) in the space.

- (a) 3 $\frac{7}{2}$ (b) -3 $-\frac{7}{2}$ (c) 3.5 $\frac{7}{2}$
- (a) $\frac{2}{3}$ 0.67 (b) $\frac{2}{3}$ -0.67 (c) $|0.67|$ $|-0.67|$

- 33–36 ■ State whether each inequality is true or false.

- (a) $-6 < -10$ (b) $\sqrt{2} > 1.41$
- (a) $\frac{10}{11} < \frac{12}{13}$ (b) $-\frac{1}{2} < -1$
- (a) $-\pi > -3$ (b) $8 \leq 9$
- (a) $1.1 > 1.\bar{1}$ (b) $8 \leq 8$

- 37–38 ■ Write each statement in terms of inequalities.

- (a) x is positive
(b) t is less than 4
(c) a is greater than or equal to π
(d) x is less than $\frac{1}{3}$ and is greater than -5
(e) The distance from p to 3 is at most 5
- (a) y is negative
(b) z is greater than 1
(c) b is at most 8
(d) w is positive and is less than or equal to 17
(e) y is at least 2 units from π

- 39–42 ■ Find the indicated set if

$$A = \{1, 2, 3, 4, 5, 6, 7\} \quad B = \{2, 4, 6, 8\}$$

$$C = \{7, 8, 9, 10\}$$

- (a) $A \cup B$ (b) $A \cap B$
- (a) $B \cup C$ (b) $B \cap C$
- (a) $A \cup C$ (b) $A \cap C$
- (a) $A \cup B \cup C$ (b) $A \cap B \cap C$

43–44 ■ Find the indicated set if

$$A = \{x \mid x \geq -2\} \quad B = \{x \mid x < 4\}$$

$$C = \{x \mid -1 < x \leq 5\}$$

43. (a) $B \cup C$ (b) $B \cap C$

44. (a) $A \cap C$ (b) $A \cap B$

45–50 ■ Express the interval in terms of inequalities, and then graph the interval.

45. $(-3, 0)$ 46. $(2, 8]$

47. $[2, 8)$ 48. $[-6, -\frac{1}{2}]$

49. $[2, \infty)$ 50. $(-\infty, 1)$

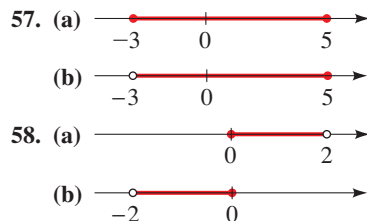
51–56 ■ Express the inequality in interval notation, and then graph the corresponding interval.

51. $x \leq 1$ 52. $1 \leq x \leq 2$

53. $-2 < x \leq 1$ 54. $x \geq -5$

55. $x > -1$ 56. $-5 < x < 2$

57–58 ■ Express each set in interval notation.



59–64 ■ Graph the set.

59. $(-2, 0) \cup (-1, 1)$ 60. $(-2, 0) \cap (-1, 1)$

61. $[-4, 6] \cap [0, 8)$ 62. $[-4, 6) \cup [0, 8)$

63. $(-\infty, -4) \cup (4, \infty)$ 64. $(-\infty, 6] \cap (2, 10)$

65–70 ■ Evaluate each expression.

65. (a) $|100|$ (b) $|-73|$

66. (a) $|\sqrt{5} - 5|$ (b) $|10 - \pi|$

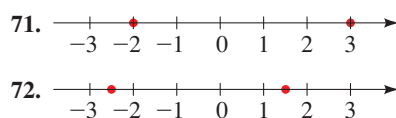
67. (a) $||-6| - |-4||$ (b) $\frac{-1}{|-1|}$

68. (a) $|2 - |-12||$ (b) $-1 - |1 - |-1||$

69. (a) $|(-2) \cdot 6|$ (b) $|(-\frac{1}{3})(-15)|$

70. (a) $|\frac{-6}{24}|$ (b) $|\frac{7-12}{12-7}|$

71–74 ■ Find the distance between the given numbers.



73. (a) 2 and 17 (b) -3 and 21 (c) $\frac{11}{8}$ and $-\frac{3}{10}$

74. (a) $\frac{7}{15}$ and $-\frac{1}{21}$ (b) -38 and -57 (c) -2.6 and -1.8

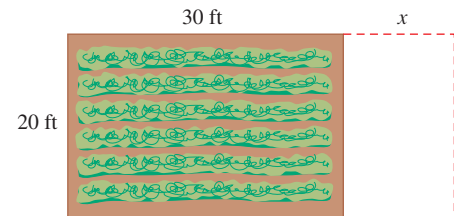
75–76 ■ Express each repeating decimal as a fraction. (See the margin note on page 2.)

75. (a) $0.\overline{7}$ (b) $0.2\overline{8}$ (c) $0.5\overline{7}$

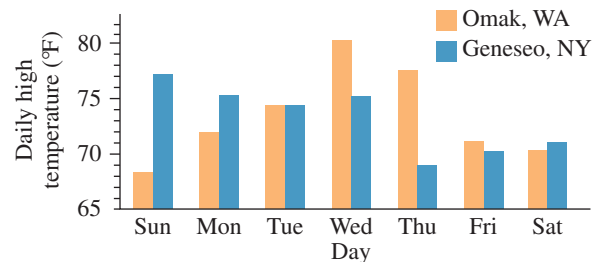
76. (a) $5.\overline{23}$ (b) $1.3\overline{7}$ (c) $2.1\overline{35}$

APPLICATIONS

77. Area of a Garden Mary's backyard vegetable garden measures 20 ft by 30 ft, so its area is $20 \times 30 = 600 \text{ ft}^2$. She decides to make it longer, as shown in the figure, so that the area increases to $A = 20(30 + x)$. Which property of real numbers tells us that the new area can also be written $A = 600 + 20x$?



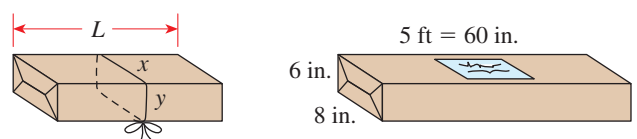
78. Temperature Variation The bar graph shows the daily high temperatures for Omak, Washington, and Geneseo, New York, during a certain week in June. Let T_O represent the temperature in Omak and T_G the temperature in Geneseo. Calculate $T_O - T_G$ and $|T_O - T_G|$ for each day shown. Which of these two values gives more information?



79. Mailing a Package The post office will only accept packages for which the length plus the "girth" (distance around) is no more than 108 inches. Thus, for the package in the figure, we must have

$$L + 2(x + y) \leq 108$$

- (a) Will the post office accept a package that is 6 in. wide, 8 in. deep, and 5 ft long? What about a package that measures 2 ft by 2 ft by 4 ft?
- (b) What is the greatest acceptable length for a package that has a square base measuring 9 in. by 9 in?



DISCOVERY ■ DISCUSSION ■ WRITING

80. Signs of Numbers Let a , b , and c be real numbers such that $a > 0$, $b < 0$, and $c < 0$. Find the sign of each expression.

- (a) $-a$ (b) $-b$ (c) bc
 (d) $a - b$ (e) $c - a$ (f) $a + bc$
 (g) $ab + ac$ (h) $-abc$ (i) ab^2

81. Sums and Products of Rational and Irrational Numbers Explain why the sum, the difference, and the product of two rational numbers are rational numbers. Is the product of two irrational numbers necessarily irrational? What about the sum?

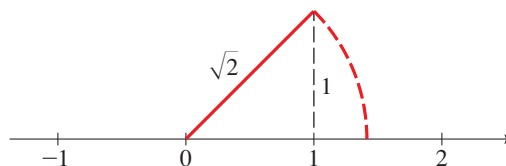
82. Combining Rational Numbers with Irrational Numbers Is $\frac{1}{2} + \sqrt{2}$ rational or irrational? Is $\frac{1}{2} \cdot \sqrt{2}$ rational or irrational? In general, what can you say about the sum of a rational and an irrational number? What about the product?

83. Limiting Behavior of Reciprocals Complete the tables. What happens to the size of the fraction $1/x$ as x gets large? As x gets small?

x	$1/x$
1	
2	
10	
100	
1000	

x	$1/x$
1.0	
0.5	
0.1	
0.01	
0.001	

84. Irrational Numbers and Geometry Using the following figure, explain how to locate the point $\sqrt{2}$ on a number line. Can you locate $\sqrt{5}$ by a similar method? What about $\sqrt{6}$? List some other irrational numbers that can be located this way.



85. Commutative and Noncommutative Operations We have seen that addition and multiplication are both commutative operations.

- (a) Is subtraction commutative?
 (b) Is division of nonzero real numbers commutative?

1.2 EXPONENTS AND RADICALS

Integer Exponents ► Rules for Working with Exponents ► Scientific Notation
 ► Radicals ► Rational Exponents ► Rationalizing the Denominator

In this section we give meaning to expressions such as $a^{m/n}$ in which the exponent m/n is a rational number. To do this, we need to recall some facts about integer exponents, radicals, and n th roots.

▼ Integer Exponents

A product of identical numbers is usually written in exponential notation. For example, $5 \cdot 5 \cdot 5$ is written as 5^3 . In general, we have the following definition.

EXPONENTIAL NOTATION

If a is any real number and n is a positive integer, then the n th power of a is

$$a^n = \underbrace{a \cdot a \cdot \cdots \cdot a}_{n \text{ factors}}$$

The number a is called the **base**, and n is called the **exponent**.

 Note the distinction between $(-3)^4$ and -3^4 . In $(-3)^4$ the exponent applies to -3 , but in -3^4 the exponent applies only to 3 .

EXAMPLE 1 | Exponential Notation

$$(a) \left(\frac{1}{2}\right)^5 = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{1}{32}$$

$$(b) (-3)^4 = (-3) \cdot (-3) \cdot (-3) \cdot (-3) = 81$$

$$(c) -3^4 = -(3 \cdot 3 \cdot 3 \cdot 3) = -81$$

 NOW TRY EXERCISE 15

We can state several useful rules for working with exponential notation. To discover the rule for multiplication, we multiply 5^4 by 5^2 :

$$5^4 \cdot 5^2 = \underbrace{(5 \cdot 5 \cdot 5 \cdot 5)}_{4 \text{ factors}} \underbrace{(5 \cdot 5)}_{2 \text{ factors}} = \underbrace{5 \cdot 5 \cdot 5 \cdot 5 \cdot 5 \cdot 5}_{6 \text{ factors}} = 5^6 = 5^{4+2}$$

It appears that *to multiply two powers of the same base, we add their exponents*. In general, for any real number a and any positive integers m and n , we have

$$a^m a^n = \underbrace{(a \cdot a \cdots a)}_{m \text{ factors}} \underbrace{(a \cdot a \cdots a)}_{n \text{ factors}} = \underbrace{a \cdot a \cdots a}_{m+n \text{ factors}} = a^{m+n}$$

Thus $a^m a^n = a^{m+n}$.

We would like this rule to be true even when m and n are 0 or negative integers. For instance, we must have

$$2^0 \cdot 2^3 = 2^{0+3} = 2^3$$

But this can happen only if $2^0 = 1$. Likewise, we want to have

$$5^4 \cdot 5^{-4} = 5^{4+(-4)} = 5^{4-4} = 5^0 = 1$$

and this will be true if $5^{-4} = 1/5^4$. These observations lead to the following definition.

ZERO AND NEGATIVE EXPONENTS

If $a \neq 0$ is any real number and n is a positive integer, then

$$a^0 = 1 \quad \text{and} \quad a^{-n} = \frac{1}{a^n}$$

EXAMPLE 2 | Zero and Negative Exponents

$$(a) \left(\frac{4}{7}\right)^0 = 1$$

$$(b) x^{-1} = \frac{1}{x^1} = \frac{1}{x}$$

$$(c) (-2)^{-3} = \frac{1}{(-2)^3} = \frac{1}{-8} = -\frac{1}{8}$$

 NOW TRY EXERCISE 17

Rules for Working with Exponents

Familiarity with the following rules is essential for our work with exponents and bases. In the table the bases a and b are real numbers, and the exponents m and n are integers.

LAWS OF EXPONENTS

Law	Example	Description
1. $a^m a^n = a^{m+n}$	$3^2 \cdot 3^5 = 3^{2+5} = 3^7$	To multiply two powers of the same number, add the exponents.
2. $\frac{a^m}{a^n} = a^{m-n}$	$\frac{3^5}{3^2} = 3^{5-2} = 3^3$	To divide two powers of the same number, subtract the exponents.
3. $(a^m)^n = a^{mn}$	$(3^2)^5 = 3^{2 \cdot 5} = 3^{10}$	To raise a power to a new power, multiply the exponents.
4. $(ab)^n = a^n b^n$	$(3 \cdot 4)^2 = 3^2 \cdot 4^2$	To raise a product to a power, raise each factor to the power.
5. $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$	$\left(\frac{3}{4}\right)^2 = \frac{3^2}{4^2}$	To raise a quotient to a power, raise both numerator and denominator to the power.

PROOF OF LAW 3 If m and n are positive integers, we have

$$\begin{aligned}
 (a^m)^n &= \underbrace{(a \cdot a \cdots a)}_{m \text{ factors}}^n \\
 &= \underbrace{(a \cdot a \cdots a)}_{m \text{ factors}} \underbrace{(a \cdot a \cdots a)}_{m \text{ factors}} \cdots \underbrace{(a \cdot a \cdots a)}_{m \text{ factors}} \\
 &\quad \underbrace{\hspace{10em}}_{n \text{ groups of factors}} \\
 &= \underbrace{a \cdot a \cdots a}_{mn \text{ factors}} = a^{mn}
 \end{aligned}$$

The cases for which $m \leq 0$ or $n \leq 0$ can be proved using the definition of negative exponents. ■

PROOF OF LAW 4 If n is a positive integer, we have

$$(ab)^n = \underbrace{(ab)(ab) \cdots (ab)}_{n \text{ factors}} = \underbrace{(a \cdot a \cdots a)}_{n \text{ factors}} \cdot \underbrace{(b \cdot b \cdots b)}_{n \text{ factors}} = a^n b^n$$

Here we have used the Commutative and Associative Properties repeatedly. If $n \leq 0$, Law 4 can be proved by using the definition of negative exponents. ■

You are asked to prove Laws 2 and 5 in Exercise 94.

EXAMPLE 3 | Using Laws of Exponents

(a) $x^4 x^7 = x^{4+7} = x^{11}$ Law 1: $a^m a^n = a^{m+n}$

(b) $y^4 y^{-7} = y^{4-7} = y^{-3} = \frac{1}{y^3}$ Law 1: $a^m a^n = a^{m+n}$

(c) $\frac{c^9}{c^5} = c^{9-5} = c^4$ Law 2: $\frac{a^m}{a^n} = a^{m-n}$

(d) $(b^4)^5 = b^{4 \cdot 5} = b^{20}$ Law 3: $(a^m)^n = a^{mn}$

(e) $(3x)^3 = 3^3 x^3 = 27x^3$ Law 4: $(ab)^n = a^n b^n$

(f) $\left(\frac{x}{2}\right)^5 = \frac{x^5}{2^5} = \frac{x^5}{32}$ Law 5: $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$

 NOW TRY EXERCISES 35, 37, AND 39 ■

EXAMPLE 4 | Simplifying Expressions with Exponents

Simplify:

$$(a) \quad (2a^3b^2)(3ab^4)^3 \qquad (b) \quad \left(\frac{x}{y}\right)^3 \left(\frac{y^2x}{z}\right)^4$$

SOLUTION

$$\begin{aligned}
 (a) \quad (2a^3b^2)(3ab^4)^3 &= (2a^3b^2)[3^3a^3(b^4)^3] && \text{Law 4: } (ab)^n = a^n b^n \\
 &= (2a^3b^2)(27a^3b^{12}) && \text{Law 3: } (a^m)^n = a^{mn} \\
 &= (2)(27)a^3a^3b^2b^{12} && \text{Group factors with the same base} \\
 &= 54a^6b^{14} && \text{Law 1: } a^m a^n = a^{m+n} \\
 (b) \quad \left(\frac{x}{y}\right)^3 \left(\frac{y^2x}{z}\right)^4 &= \frac{x^3}{y^3} \frac{(y^2)^4 x^4}{z^4} && \text{Laws 5 and 4} \\
 &= \frac{x^3}{y^3} \frac{y^8 x^4}{z^4} && \text{Law 3} \\
 &= (x^3 x^4) \left(\frac{y^8}{y^3}\right) \frac{1}{z^4} && \text{Group factors with the same base} \\
 &= \frac{x^7 y^5}{z^4} && \text{Laws 1 and 2}
 \end{aligned}$$

 NOW TRY EXERCISES 43 AND 47

When simplifying an expression, you will find that many different methods will lead to the same result; you should feel free to use any of the rules of exponents to arrive at your own method. We now give two additional laws that are useful in simplifying expressions with negative exponents.

LAWS OF EXPONENTS

Law	Example	Description
6. $\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^n$	$\left(\frac{3}{4}\right)^{-2} = \left(\frac{4}{3}\right)^2$	To raise a fraction to a negative power, invert the fraction and change the sign of the exponent.
7. $\frac{a^{-n}}{b^{-m}} = \frac{b^m}{a^n}$	$\frac{3^{-2}}{4^{-5}} = \frac{4^5}{3^2}$	To move a number raised to a power from numerator to denominator or from denominator to numerator, change the sign of the exponent.

PROOF OF LAW 7 Using the definition of negative exponents and then Property 2 of fractions (page 5), we have

$$\frac{a^{-n}}{b^{-m}} = \frac{1/a^n}{1/b^m} = \frac{1}{a^n} \cdot \frac{b^m}{1} = \frac{b^m}{a^n}$$

You are asked to prove Law 6 in Exercise 94.

EXAMPLE 5 | Simplifying Expressions with Negative Exponents

Eliminate negative exponents and simplify each expression.

$$(a) \quad \frac{6st^{-4}}{2s^{-2}t^2} \qquad (b) \quad \left(\frac{y}{3z^3}\right)^{-2}$$

MATHEMATICS IN THE MODERN WORLD

Although we are often unaware of its presence, mathematics permeates nearly every aspect of life in the modern world. With the advent of modern technology, mathematics plays an ever greater role in our lives. Today you were probably awakened by a digital alarm clock, made a phone call that used digital transmission, sent an e-mail message over the Internet, drove a car with digitally controlled fuel injection, listened to music on a CD player or MP3 player, perhaps watched a DVD or digital television, then fell asleep in a room whose temperature is controlled by a digital thermostat. In each of these activities mathematics is crucially involved. In general, a property such as the intensity or frequency of sound, the oxygen level in the exhaust emission from a car, the colors in an image, or the temperature in your bedroom is transformed into sequences of numbers by sophisticated mathematical algorithms. These numerical data, which usually consist of many millions of bits (the digits 0 and 1), are then transmitted and reinterpreted. Dealing with such huge amounts of data was not feasible until the invention of computers, machines whose logical processes were invented by mathematicians.

The contributions of mathematics in the modern world are not limited to technological advances. The logical processes of mathematics are now used to analyze complex problems in the social, political, and life sciences in new and surprising ways. Advances in mathematics continue to be made, some of the most exciting of these just within the past decade.

In other *Mathematics in the Modern World*, we will describe in more detail how mathematics affects all of us in our everyday activities.

SOLUTION

- (a) We use Law 7, which allows us to move a number raised to a power from the numerator to the denominator (or vice versa) by changing the sign of the exponent.

$$\begin{aligned} \frac{6st^{-4}}{2s^{-2}t^2} &= \frac{6s^2}{2t^2t^4} && \text{Law 7} \\ &= \frac{3s^2}{t^6} && \text{Law 1} \end{aligned}$$

t⁻⁴ moves to denominator and becomes *t*⁴

s⁻² moves to numerator and becomes *s*²

- (b) We use Law 6, which allows us to change the sign of the exponent of a fraction by inverting the fraction.

$$\begin{aligned} \left(\frac{y}{3z^3}\right)^{-2} &= \left(\frac{3z^3}{y}\right)^2 && \text{Law 6} \\ &= \frac{9z^6}{y^2} && \text{Laws 5 and 4} \end{aligned}$$

NOW TRY EXERCISE 49

Scientific Notation

Scientists use exponential notation as a compact way of writing very large numbers and very small numbers. For example, the nearest star beyond the sun, Proxima Centauri, is approximately 40,000,000,000,000 km away. The mass of a hydrogen atom is about 0.000000000000000000000000166 g. Such numbers are difficult to read and to write, so scientists usually express them in *scientific notation*.

SCIENTIFIC NOTATION

A positive number x is said to be written in **scientific notation** if it is expressed as follows:

$$x = a \times 10^n \quad \text{where } 1 \leq a < 10 \text{ and } n \text{ is an integer}$$

For instance, when we state that the distance to the star Proxima Centauri is 4×10^{13} km, the positive exponent 13 indicates that the decimal point should be moved 13 places to the *right*:

$$4 \times 10^{13} = 40,000,000,000,000$$

Move decimal point 13 places to the right

When we state that the mass of a hydrogen atom is 1.66×10^{-24} g, the exponent -24 indicates that the decimal point should be moved 24 places to the *left*:

$$1.66 \times 10^{-24} = 0.000000000000000000000000166$$

Move decimal point 24 places to the left

EXAMPLE 6 | Changing from Decimal to Scientific Notation

Write each number in scientific notation.

(a) 56,920

(b) 0.000093

SOLUTION

(a) $56,920 = 5.692 \times 10^4$
4 places

(b) $0.000093 = 9.3 \times 10^{-5}$
5 places

 NOW TRY EXERCISES 77 AND 79

To use scientific notation on a calculator, press the key labeled **EE** or **EXP** or **EE $\pm$** to enter the exponent. For example, to enter the number 3.629×10^{15} on a TI-83 calculator, we enter

3.629 **2ND** **EE** 15

and the display reads

3.629E15

Scientific notation is often used on a calculator to display a very large or very small number. For instance, if we use a calculator to square the number 1,111,111, the display panel may show (depending on the calculator model) the approximation

$$1.234568 \ 12 \quad \text{or} \quad 1.23468 \ E12$$

Here the final digits indicate the power of 10, and we interpret the result as

$$1.234568 \times 10^{12}$$

EXAMPLE 7 | Calculating with Scientific Notation

If $a \approx 0.00046$, $b \approx 1.697 \times 10^{22}$, and $c \approx 2.91 \times 10^{-18}$, use a calculator to approximate the quotient ab/c .

SOLUTION We could enter the data using scientific notation, or we could use laws of exponents as follows:

$$\begin{aligned} \frac{ab}{c} &\approx \frac{(4.6 \times 10^{-4})(1.697 \times 10^{22})}{2.91 \times 10^{-18}} \\ &= \frac{(4.6)(1.697)}{2.91} \times 10^{-4+22+18} \\ &\approx 2.7 \times 10^{36} \end{aligned}$$

We state the answer rounded to two significant figures because the least accurate of the given numbers is stated to two significant figures.

 NOW TRY EXERCISES 83 AND 85**Radicals**

We know what 2^n means whenever n is an integer. To give meaning to a power, such as $2^{4/5}$, whose exponent is a rational number, we need to discuss radicals.

The symbol $\sqrt{}$ means “the positive square root of.” Thus

$$\sqrt{a} = b \quad \text{means} \quad b^2 = a \quad \text{and} \quad b \geq 0$$

Since $a = b^2 \geq 0$, the symbol $\sqrt{a}$ makes sense only when $a \geq 0$. For instance,

$$\sqrt{9} = 3 \quad \text{because} \quad 3^2 = 9 \quad \text{and} \quad 3 \geq 0$$

For guidelines on working with significant figures, see the Appendix, *Calculations and Significant Figures*.

It is true that the number 9 has two square roots, 3 and -3 , but the notation $\sqrt{9}$ is reserved for the *positive* square root of 9 (sometimes called the *principal square root* of 9). If we want the negative root, we must write $-\sqrt{9}$, which is -3 .

Square roots are special cases of n th roots. The n th root of x is the number that, when raised to the n th power, gives x .

DEFINITION OF n th ROOT

If n is any positive integer, then the **principal n th root** of a is defined as follows:

$$\sqrt[n]{a} = b \quad \text{means} \quad b^n = a$$

If n is even, we must have $a \geq 0$ and $b \geq 0$.

Thus

$$\begin{aligned} \sqrt[4]{81} &= 3 & \text{because} & \quad 3^4 = 81 & \text{and} & \quad 3 \geq 0 \\ \sqrt[3]{-8} &= -2 & \text{because} & \quad (-2)^3 = -8 \end{aligned}$$

But $\sqrt{-8}$, $\sqrt[4]{-8}$, and $\sqrt[6]{-8}$ are not defined. (For instance, $\sqrt{-8}$ is not defined because the square of every real number is nonnegative.)

Notice that

$$\sqrt{4^2} = \sqrt{16} = 4 \quad \text{but} \quad \sqrt{(-4)^2} = \sqrt{16} = 4 = |-4|$$

So the equation $\sqrt{a^2} = a$ is not always true; it is true only when $a \geq 0$. However, we can always write $\sqrt{a^2} = |a|$. This last equation is true not only for square roots, but for any even root. This and other rules used in working with n th roots are listed in the following box. In each property we assume that all the given roots exist.

PROPERTIES OF n th ROOTS

Property	Example
1. $\sqrt[n]{ab} = \sqrt[n]{a}\sqrt[n]{b}$	$\sqrt[3]{-8 \cdot 27} = \sqrt[3]{-8}\sqrt[3]{27} = (-2)(3) = -6$
2. $\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$	$\sqrt[4]{\frac{16}{81}} = \frac{\sqrt[4]{16}}{\sqrt[4]{81}} = \frac{2}{3}$
3. $\sqrt[m]{\sqrt[n]{a}} = \sqrt[mn]{a}$	$\sqrt{\sqrt[3]{729}} = \sqrt[6]{729} = 3$
4. $\sqrt[n]{a^n} = a$ if n is odd	$\sqrt[3]{(-5)^3} = -5, \quad \sqrt[5]{2^5} = 2$
5. $\sqrt[n]{a^n} = a $ if n is even	$\sqrt[4]{(-3)^4} = -3 = 3$

EXAMPLE 8 | Simplifying Expressions Involving n th Roots

- (a) $\sqrt[3]{x^4} = \sqrt[3]{x^3x}$ Factor out the largest cube
 $= \sqrt[3]{x^3}\sqrt[3]{x}$ Property 1: $\sqrt[3]{ab} = \sqrt[3]{a}\sqrt[3]{b}$
 $= x\sqrt[3]{x}$ Property 4: $\sqrt[3]{a^3} = a$
- (b) $\sqrt[4]{81x^8y^4} = \sqrt[4]{81}\sqrt[4]{x^8}\sqrt[4]{y^4}$ Property 1: $\sqrt[4]{abc} = \sqrt[4]{a}\sqrt[4]{b}\sqrt[4]{c}$
 $= 3\sqrt[4]{(x^2)^4}|y|$ Property 5: $\sqrt[4]{a^4} = |a|$
 $= 3x^2|y|$ Property 5: $\sqrt[4]{a^4} = |a|, |x^2| = x^2$

 **NOW TRY EXERCISES 55 AND 57**

It is frequently useful to combine like radicals in an expression such as $2\sqrt{3} + 5\sqrt{3}$. This can be done by using the Distributive Property. Thus

$$2\sqrt{3} + 5\sqrt{3} = (2 + 5)\sqrt{3} = 7\sqrt{3}$$

The next example further illustrates this process.

EXAMPLE 9 | Combining Radicals

 Avoid making the following error:

$$\sqrt{a+b} \neq \sqrt{a} + \sqrt{b}$$

For instance, if we let $a = 9$ and $b = 16$, then we see the error:

$$\sqrt{9+16} \neq \sqrt{9} + \sqrt{16}$$

$$\sqrt{25} \neq 3 + 4$$

$$5 \neq 7 \quad \text{Wrong!}$$

$$(a) \quad \sqrt{32} + \sqrt{200} = \sqrt{16 \cdot 2} + \sqrt{100 \cdot 2}$$

$$= \sqrt{16}\sqrt{2} + \sqrt{100}\sqrt{2}$$

$$= 4\sqrt{2} + 10\sqrt{2} = 14\sqrt{2}$$

Factor out the largest squares

Property 1: $\sqrt{ab} = \sqrt{a}\sqrt{b}$

Distributive Property

$$(b) \quad \text{If } b > 0, \text{ then}$$

$$\sqrt{25b} - \sqrt{b^3} = \sqrt{25}\sqrt{b} - \sqrt{b^2}\sqrt{b}$$

$$= 5\sqrt{b} - b\sqrt{b}$$

$$= (5 - b)\sqrt{b}$$

Property 1: $\sqrt{ab} = \sqrt{a}\sqrt{b}$

Property 5, $b > 0$

Distributive Property

NOW TRY EXERCISES 29 AND 33

▼ Rational Exponents

To define what is meant by a *rational exponent* or, equivalently, a *fractional exponent* such as $a^{1/3}$, we need to use radicals. To give meaning to the symbol $a^{1/n}$ in a way that is consistent with the Laws of Exponents, we would have to have

$$(a^{1/n})^n = a^{(1/n)n} = a^1 = a$$

So by the definition of n th root,

$$a^{1/n} = \sqrt[n]{a}$$

In general, we define rational exponents as follows.

DEFINITION OF RATIONAL EXPONENTS

For any rational exponent m/n in lowest terms, where m and n are integers and $n > 0$, we define

$$a^{m/n} = (\sqrt[n]{a})^m \quad \text{or equivalently} \quad a^{m/n} = \sqrt[n]{a^m}$$

If n is even, then we require that $a \geq 0$.

With this definition it can be proved that *the Laws of Exponents also hold for rational exponents*.

EXAMPLE 10 | Using the Definition of Rational Exponents

$$(a) \quad 4^{1/2} = \sqrt{4} = 2$$

$$(b) \quad 8^{2/3} = (\sqrt[3]{8})^2 = 2^2 = 4 \quad \text{Alternative solution: } 8^{2/3} = \sqrt[3]{8^2} = \sqrt[3]{64} = 4$$

$$(c) \quad 125^{-1/3} = \frac{1}{125^{1/3}} = \frac{1}{\sqrt[3]{125}} = \frac{1}{5} \quad (d) \quad \frac{1}{\sqrt[3]{x^4}} = \frac{1}{x^{4/3}} = x^{-4/3}$$

NOW TRY EXERCISES 21 AND 23

DIOPHANTUS lived in Alexandria about 250 A.D. His book *Arithmetica* is considered the first book on algebra. In it he gives methods for finding integer solutions of algebraic equations. *Arithmetica* was read and studied for more than a thousand years. Fermat (see page 99) made some of his most important discoveries while studying this book. Diophantus' major contribution is the use of symbols to stand for the unknowns in a problem. Although his symbolism is not as simple as what we use today, it was a major advance over writing everything in words. In Diophantus' notation the equation

$$x^5 - 7x^2 + 8x - 5 = 24$$

is written

$$\Delta K^{\gamma} \alpha \varsigma \eta \text{ } \eta^{\flat} \Delta^{\gamma} \zeta^{\circ} M \epsilon \iota^{\sigma} \kappa \delta$$

Our modern algebraic notation did not come into common use until the 17th century.

EXAMPLE 11 | Using the Laws of Exponents with Rational Exponents

- (a) $a^{1/3} a^{7/3} = a^{8/3}$ Law 1: $a^m a^n = a^{m+n}$
- (b) $\frac{a^{2/5} a^{7/5}}{a^{3/5}} = a^{2/5 + 7/5 - 3/5} = a^{6/5}$ Law 1, Law 2: $\frac{a^m}{a^n} = a^{m-n}$
- (c) $(2a^3 b^4)^{3/2} = 2^{3/2} (a^3)^{3/2} (b^4)^{3/2}$ Law 4: $(abc)^n = a^n b^n c^n$
 $= (\sqrt{2})^3 a^{3(3/2)} b^{4(3/2)}$ Law 3: $(a^m)^n = a^{mn}$
 $= 2\sqrt{2} a^{9/2} b^6$
- (d) $\left(\frac{2x^{3/4}}{y^{1/3}}\right)^3 \left(\frac{y^4}{x^{-1/2}}\right) = \frac{2^3 (x^{3/4})^3}{(y^{1/3})^3} \cdot (y^4 x^{1/2})$ Laws 5, 4, and 7
 $= \frac{8x^{9/4}}{y} \cdot y^4 x^{1/2}$ Law 3
 $= 8x^{11/4} y^3$ Laws 1 and 2

 **NOW TRY EXERCISES 61, 63, 67, AND 69**

EXAMPLE 12 | Simplifying by Writing Radicals as Rational Exponents

- (a) $(2\sqrt{x})(3\sqrt[3]{x}) = (2x^{1/2})(3x^{1/3})$ Definition of rational exponents
 $= 6x^{1/2 + 1/3} = 6x^{5/6}$ Law 1
- (b) $\sqrt{x}\sqrt{x} = (xx^{1/2})^{1/2}$ Definition of rational exponents
 $= (x^{3/2})^{1/2}$ Law 1
 $= x^{3/4}$ Law 3

 **NOW TRY EXERCISES 71 AND 75**

▼ Rationalizing the Denominator

It is often useful to eliminate the radical in a denominator by multiplying both numerator and denominator by an appropriate expression. This procedure is called **rationalizing the denominator**. If the denominator is of the form $\sqrt{a}$, we multiply numerator and denominator by $\sqrt{a}$. In doing this we multiply the given quantity by 1, so we do not change its value. For instance,

$$\frac{1}{\sqrt{a}} = \frac{1}{\sqrt{a}} \cdot 1 = \frac{1}{\sqrt{a}} \cdot \frac{\sqrt{a}}{\sqrt{a}} = \frac{\sqrt{a}}{a}$$

Note that the denominator in the last fraction contains no radical. In general, if the denominator is of the form $\sqrt[n]{a^m}$ with $m < n$, then multiplying the numerator and denominator by $\sqrt[n]{a^{n-m}}$ will rationalize the denominator, because (for $a > 0$)

$$\sqrt[n]{a^m} \sqrt[n]{a^{n-m}} = \sqrt[n]{a^{m+n-m}} = \sqrt[n]{a^n} = a$$

EXAMPLE 13 | Rationalizing Denominators

- (a) $\frac{2}{\sqrt{3}} = \frac{2}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$ This equals 1

$$(b) \frac{1}{\sqrt[3]{x^2}} = \frac{1}{\sqrt[3]{x^2}} \frac{\sqrt[3]{x}}{\sqrt[3]{x}} = \frac{\sqrt[3]{x}}{\sqrt[3]{x^3}} = \frac{\sqrt[3]{x}}{x}$$

$$(c) \sqrt[7]{\frac{1}{a^2}} = \frac{1}{\sqrt[7]{a^2}} = \frac{1}{\sqrt[7]{a^2}} \frac{\sqrt[7]{a^5}}{\sqrt[7]{a^5}} = \frac{\sqrt[7]{a^5}}{\sqrt[7]{a^7}} = \frac{\sqrt[7]{a^5}}{a}$$

 **NOW TRY EXERCISES 89 AND 91**

1.2 EXERCISES

CONCEPTS

- (a) Using exponential notation, we can write the product $5 \cdot 5 \cdot 5 \cdot 5 \cdot 5 \cdot 5$ as _____.
(b) In the expression 3^4 , the number 3 is called the _____, and the number 4 is called the _____.
- (a) When we multiply two powers with the same base, we _____ the exponents. So $3^4 \cdot 3^5 =$ _____.
(b) When we divide two powers with the same base, we _____ the exponents. So $\frac{3^5}{3^2} =$ _____.
- (a) Using exponential notation, we can write $\sqrt[3]{5}$ as _____.
(b) Using radicals, we can write $5^{1/2}$ as _____.
(c) Is there a difference between $\sqrt{5^2}$ and $(\sqrt{5})^2$? Explain.
- Explain what $4^{3/2}$ means, then calculate $4^{3/2}$ in two different ways:
 $(4^{1/2})^3 =$ _____ or $(4^3)^{1/2} =$ _____
- Explain how we rationalize a denominator, then complete the following steps to rationalize $\frac{1}{\sqrt{3}}$:
$$\frac{1}{\sqrt{3}} = \frac{1}{\sqrt{3}} \cdot \frac{\square}{\square} = \frac{\square}{\square}$$
- Find the missing power in the following calculation:
 $5^{1/3} \cdot 5^{\square} = 5.$

SKILLS

7–14 ■ Write each radical expression using exponents, and each exponential expression using radicals.

	Radical expression	Exponential expression
7.	$\frac{1}{\sqrt{5}}$	
8.	$\sqrt[3]{7^2}$	
9.		$4^{2/3}$
10.		$11^{-3/2}$
11.	$\sqrt[5]{5^3}$	
12.		$2^{-1.5}$

	Radical expression	Exponential expression
13.		$a^{2/5}$
14.	$\frac{1}{\sqrt{x^5}}$	

15–24 ■ Evaluate each expression.

- | | | |
|---|---|--|
|  15. (a) -3^2 | (b) $(-3)^2$ | (c) $(\frac{1}{3})^4(-3)^2$ |
| 16. (a) $5^4 \cdot 5^{-2}$ | (b) $\frac{10^7}{10^4}$ | (c) $\frac{3}{3^{-2}}$ |
|  17. (a) $(\frac{5}{3})^0 2^{-1}$ | (b) $\frac{2^{-3}}{3^0}$ | (c) $(\frac{1}{4})^{-2}$ |
| 18. (a) $(-\frac{2}{3})^{-3}$ | (b) $(\frac{3}{2})^{-2} \cdot \frac{9}{16}$ | (c) $(\frac{1}{2})^4 \cdot (\frac{5}{2})^{-2}$ |
| 19. (a) $\sqrt{16}$ | (b) $\sqrt[4]{16}$ | (c) $\sqrt[4]{\frac{1}{16}}$ |
| 20. (a) $\sqrt[3]{64}$ | (b) $\sqrt[3]{-64}$ | (c) $\sqrt[5]{-32}$ |
|  21. (a) $\sqrt[4]{\frac{4}{9}}$ | (b) $\sqrt[4]{256}$ | (c) $\sqrt[6]{\frac{1}{64}}$ |
| 22. (a) $\sqrt{7}\sqrt{28}$ | (b) $\frac{\sqrt{48}}{\sqrt{3}}$ | (c) $\sqrt[4]{24}\sqrt[4]{54}$ |
|  23. (a) $(\frac{4}{9})^{-1/2}$ | (b) $(-32)^{2/5}$ | (c) $-32^{2/5}$ |
| 24. (a) $1024^{-0.1}$ | (b) $(-\frac{27}{8})^{2/3}$ | (c) $(\frac{25}{64})^{-3/2}$ |

25–28 ■ Evaluate the expression using $x = 3$, $y = 4$, and $z = -1$.

- | | |
|---|--------------------------------|
| 25. $\sqrt{x^2 + y^2}$ | 26. $\sqrt[4]{x^3 + 14y} + 2z$ |
| 27. $(9x)^{2/3} + (2y)^{2/3} + z^{2/3}$ | 28. $(xy)^{2z}$ |

29–34 ■ Simplify the expression.

- | | |
|---|------------------------------------|
|  29. $\sqrt{32} + \sqrt{18}$ | 30. $\sqrt{75} + \sqrt{48}$ |
| 31. $\sqrt[5]{96} + \sqrt[5]{3}$ | 32. $\sqrt[4]{48} - \sqrt[4]{3}$ |
|  33. $\sqrt{16x} + \sqrt{x^5}$ | 34. $\sqrt[3]{2y^4} - \sqrt[3]{y}$ |

35–40 ■ Simplify each expression.

- | | | |
|--|--------------------------|----------------------------|
|  35. (a) $x^8 x^2$ | (b) $(3y^2)(4y^5)$ | (c) $x^2 x^{-6}$ |
| 36. (a) $x^{-5} x^3$ | (b) $w^{-2} w^{-4} w^6$ | (c) $z^5 z^{-3} z^{-4}$ |
|  37. (a) $\frac{y^{10} y^0}{y^7}$ | (b) $\frac{x^6}{x^{10}}$ | (c) $\frac{a^9 a^{-2}}{a}$ |
| 38. (a) $\frac{z^2 z^4}{z^3 z^{-1}}$ | (b) $(2y^2)^3$ | (c) $(8x)^2$ |

39. (a) $(a^2a^4)^3$ (b) $\left(\frac{a^2}{4}\right)^3$ (c) $(3z)^2(6z^2)^{-3}$

40. (a) $(2z^2)^{-5}z^{10}$ (b) $(2a^3a^2)^4$ (c) $\left(\frac{3x^4}{4x^2}\right)^2$

41–52 ■ Simplify the expression and eliminate any negative exponents(s).

41. (a) $(4x^2y^4)(2x^5y)$ (b) $(8a^2z)(\frac{1}{2}a^3z^4)$

42. (a) $b^4(3ab^3)(2a^2b^{-5})$ (b) $(2s^3t^{-2})(\frac{1}{4}s^7t)(16t^4)$

43. (a) $(5x^2y^3)(3x^2y^5)^4$ (b) $(2a^3b^2)^2(5a^2b^5)^3$

44. (a) $(s^{-2}t^2)^2(s^2t)^3$ (b) $(2u^2v^3)^3(3u^{-3}v)^2$

45. (a) $\frac{6y^3z}{2yz^2}$ (b) $\frac{(xy^2z^3)^4}{(x^2y^2z)^3}$

46. (a) $\frac{2x^3y^4}{x^5y^3}$ (b) $\frac{(2v^3w)^2}{v^3w^2}$

47. (a) $\left(\frac{a^2}{b}\right)^5\left(\frac{a^3b^2}{c^3}\right)^3$ (b) $\frac{(u^{-1}v^2)^2}{(u^3v^{-2})^3}$

48. (a) $\left(\frac{x^4z^2}{4y^5}\right)\left(\frac{2x^3y^2}{z^3}\right)^2$ (b) $\frac{(rs^2)^3}{(r^{-3}s^2)^2}$

49. (a) $\frac{8a^3b^{-4}}{2a^{-5}b^5}$ (b) $\left(\frac{y}{5x^{-2}}\right)^{-3}$

50. (a) $\frac{5xy^{-2}}{x^{-1}y^{-3}}$ (b) $\left(\frac{2a^{-1}b}{a^2b^{-3}}\right)^{-3}$

51. (a) $\left(\frac{3a}{b^3}\right)^{-1}$ (b) $\left(\frac{q^{-1}r^{-1}s^{-2}}{r^{-5}sq^{-8}}\right)^{-1}$

52. (a) $\left(\frac{s^2t^{-4}}{5s^{-1}t}\right)^{-2}$ (b) $\left(\frac{xy^{-2}z^{-3}}{x^2y^3z^{-4}}\right)^{-3}$

53–60 ■ Simplify the expression. Assume that the letters denote any real numbers.

53. $\sqrt[4]{x^4}$ 54. $\sqrt[5]{x^{10}}$

55. $\sqrt[4]{16x^8}$ 56. $\sqrt[3]{x^3y^6}$

57. $\sqrt[6]{64a^6b^7}$ 58. $\sqrt[3]{a^2b^3}\sqrt[3]{64d^4b}$

59. $\sqrt[3]{\sqrt{64x^6}}$ 60. $\sqrt[4]{x^4y^2z^2}$

61–70 ■ Simplify the expression and eliminate any negative exponent(s). Assume that all letters denote positive numbers.

61. (a) $x^{3/4}x^{5/4}$ (b) $y^{2/3}y^{4/3}$

62. (a) $(4b)^{1/2}(8b^{1/4})$ (b) $(3a^{3/4})^2(5a^{1/2})$

63. (a) $\frac{w^{4/3}w^{2/3}}{w^{1/3}}$ (b) $\frac{s^{5/2}(2s^{5/4})^2}{s^{1/2}}$

64. (a) $(8y^3)^{-2/3}$ (b) $(u^4v^6)^{-1/3}$

65. (a) $(8a^6b^{3/2})^{2/3}$ (b) $(4a^6b^8)^{3/2}$

66. (a) $(x^{-5}y^{1/3})^{-3/5}$ (b) $(2x^3y^{-1/4})^2(8y^{-3/2})^{-1/3}$

67. (a) $\frac{(8s^3t^3)^{2/3}}{(s^4t^{-8})^{1/4}}$ (b) $\frac{(32y^{-5}z^{10})^{1/5}}{(64y^6z^{-12})^{-1/6}}$

68. (a) $\left(\frac{x^8y^{-4}}{16y^{4/3}}\right)^{-1/4}$ (b) $\left(\frac{-8y^{3/4}}{y^3z^6}\right)^{-1/3}$

69. (a) $\left(\frac{x^{-2/3}}{y^{1/2}}\right)\left(\frac{x^{-2}}{y^{-3}}\right)^{1/6}$ (b) $\left(\frac{4y^3z^{2/3}}{x^{1/2}}\right)^2\left(\frac{x^{-3}y^6}{8z^4}\right)^{1/3}$

70. (a) $\left(\frac{a^{1/6}b^{-3}}{x^{-1}y}\right)^3\left(\frac{x^{-2}b^{-1}}{a^{3/2}y^{1/3}}\right)$ (b) $\frac{(9st)^{3/2}}{(27s^3t^{-4})^{2/3}}\left(\frac{3s^{-2}}{4t^{1/3}}\right)^{-1}$

71–76 ■ Simplify the expression and eliminate any negative exponents(s). Assume that all letters denote positive numbers.

71. (a) $\sqrt[6]{y^5}\sqrt[3]{y^2}$ (b) $(5\sqrt[3]{x})(2\sqrt[4]{x})$

72. (a) $\sqrt[4]{b^3}\sqrt{b}$ (b) $(2\sqrt{a})(\sqrt[3]{a^2})$

73. (a) $\sqrt{4st^3}\sqrt[6]{s^3t^2}$ (b) $\frac{\sqrt[4]{x^7}}{\sqrt[4]{x^3}}$

74. (a) $\sqrt[5]{x^3y^2}\sqrt[10]{x^4y^{16}}$ (b) $\frac{\sqrt[3]{8x^2}}{\sqrt{x}}$

75. (a) $\sqrt[3]{y}\sqrt{y}$ (b) $\sqrt{\frac{16u^3v}{uv^5}}$

76. (a) $\sqrt{s}\sqrt{s^3}$ (b) $\sqrt[3]{\frac{54x^2y^4}{2x^5y}}$

77–78 ■ Write each number in scientific notation.

77. (a) 69,300,000 (b) 7,200,000,000,000

(c) 0.000028536 (d) 0.0001213

78. (a) 129,540,000 (b) 7,259,000,000

(c) 0.0000000014 (d) 0.0007029

79–80 ■ Write each number in decimal notation.

79. (a) 3.19×10^5 (b) 2.721×10^8

(c) 2.670×10^{-8} (d) 9.999×10^{-9}

80. (a) 7.1×10^{14} (b) 6×10^{12}

(c) 8.55×10^{-3} (d) 6.257×10^{-10}

81–82 ■ Write the number indicated in each statement in scientific notation.

81. (a) A light-year, the distance that light travels in one year, is about 5,900,000,000,000 mi.
- (b) The diameter of an electron is about 0.0000000000004 cm.
- (c) A drop of water contains more than 33 billion billion molecules.
82. (a) The distance from the earth to the sun is about 93 million miles.
- (b) The mass of an oxygen molecule is about 0.000000000000000000000053 g.
- (c) The mass of the earth is about 5,970,000,000,000,000,000,000,000 kg.

83–88 ■ Use scientific notation, the Laws of Exponents, and a calculator to perform the indicated operations. State your answer rounded to the number of significant digits indicated by the given data.

83. $(7.2 \times 10^{-9})(1.806 \times 10^{-12})$

84. $(1.062 \times 10^{24})(8.61 \times 10^{19})$

85. $\frac{1.295643 \times 10^9}{(3.610 \times 10^{-17})(2.511 \times 10^6)}$

86. $\frac{(73.1)(1.6341 \times 10^{28})}{0.0000000019}$

87. $\frac{(0.0000162)(0.01582)}{(594,621,000)(0.0058)}$

88. $\frac{(3.542 \times 10^{-6})^9}{(5.05 \times 10^4)^{12}}$

89–92 ■ Rationalize the denominator.

89. (a) $\frac{1}{\sqrt{10}}$ (b) $\sqrt{\frac{2}{x}}$ (c) $\sqrt{\frac{x}{3}}$

90. (a) $\sqrt{\frac{5}{12}}$ (b) $\sqrt{\frac{x}{6}}$ (c) $\sqrt{\frac{y}{2z}}$

91. (a) $\frac{2}{\sqrt[3]{x}}$ (b) $\frac{1}{\sqrt[4]{y^3}}$ (c) $\frac{x}{y^{2/5}}$

92. (a) $\frac{1}{\sqrt[4]{a}}$ (b) $\frac{a}{\sqrt[3]{b^2}}$ (c) $\frac{1}{c^{3/7}}$

93. Let a , b , and c be real numbers with $a > 0$, $b < 0$, and $c < 0$. Determine the sign of each expression.

(a) b^5 (b) b^{10} (c) ab^2c^3

(d) $(b - a)^3$ (e) $(b - a)^4$ (f) $\frac{a^3c^3}{b^6c^6}$

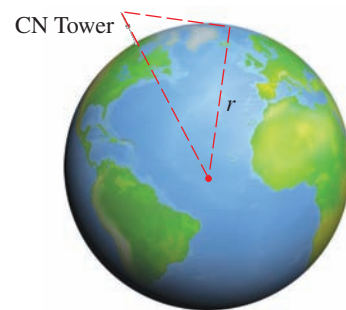
94. Prove the given Laws of Exponents for the case in which m and n are positive integers and $m > n$.

(a) Law 2 (b) Law 5 (c) Law 6

APPLICATIONS

95. **Distance to the Nearest Star** Proxima Centauri, the star nearest to our solar system, is 4.3 light-years away. Use the information in Exercise 81(a) to express this distance in miles.96. **Speed of Light** The speed of light is about 186,000 mi/s. Use the information in Exercise 82(a) to find how long it takes for a light ray from the sun to reach the earth.97. **Volume of the Oceans** The average ocean depth is 3.7×10^3 m, and the area of the oceans is 3.6×10^{14} m². What is the total volume of the ocean in liters? (One cubic meter contains 1000 liters.)98. **National Debt** As of July 2010, the population of the United States was 3.070×10^8 , and the national debt was 1.320×10^{13} dollars. How much was each person's share of the debt?99. **Number of Molecules** A sealed room in a hospital, measuring 5 m wide, 10 m long, and 3 m high, is filled with pure oxygen. One cubic meter contains 1000 L, and 22.4 L of any gas contains 6.02×10^{23} molecules (Avogadro's number). How many molecules of oxygen are there in the room?100. **How Far Can You See?** Because of the curvature of the earth, the maximum distance D that you can see from the top of a tall building of height h is estimated by the formula

$$D = \sqrt{2rh + h^2}$$

where $r = 3960$ mi is the radius of the earth and D and h are also measured in miles. How far can you see from the observation deck of the Toronto CN Tower, 1135 ft above the ground?101. **Speed of a Skidding Car** Police use the formula $s = \sqrt{30fd}$ to estimate the speed s (in mi/h) at which a car is traveling if it skids d feet after the brakes are applied suddenly. The number f is the coefficient of friction of the road, which is a measure of the "slipperiness" of the road. The table gives some typical estimates for f .

	Tar	Concrete	Gravel
Dry	1.0	0.8	0.2
Wet	0.5	0.4	0.1

- (a) If a car skids 65 ft on wet concrete, how fast was it moving when the brakes were applied?
- (b) If a car is traveling at 50 mi/h, how far will it skid on wet tar?



- 102. Distance from the Earth to the Sun** It follows from **Kepler's Third Law** of planetary motion that the average distance from a planet to the sun (in meters) is

$$d = \left(\frac{GM}{4\pi^2} \right)^{1/3} T^{2/3}$$

where $M = 1.99 \times 10^{30}$ kg is the mass of the sun, $G = 6.67 \times 10^{-11}$ N·m²/kg² is the gravitational constant, and T is the period of the planet's orbit (in seconds). Use the fact that the period of the earth's orbit is about 365.25 days to find the distance from the earth to the sun.

DISCOVERY ■ DISCUSSION ■ WRITING

- 103. How Big Is a Billion?** If you had a million (10^6) dollars in a suitcase, and you spent a thousand (10^3) dollars each day, how many years would it take you to use all the money? Spending at the same rate, how many years would it take you to empty a suitcase filled with a *billion* (10^9) dollars?

- 104. Easy Powers That Look Hard** Calculate these expressions in your head. Use the Laws of Exponents to help you.

(a) $\frac{18^5}{9^5}$ (b) $20^6 \cdot (0.5)^6$

- 105. Limiting Behavior of Powers** Complete the following tables. What happens to the n th root of 2 as n gets large? What about the n th root of $\frac{1}{2}$?

n	$2^{1/n}$
1	
2	
5	
10	
100	

n	$(\frac{1}{2})^{1/n}$
1	
2	
5	
10	
100	

Construct a similar table for $n^{1/n}$. What happens to the n th root of n as n gets large?

- 106. Comparing Roots** Without using a calculator, determine which number is larger in each pair.

(a) $2^{1/2}$ or $2^{1/3}$ (b) $(\frac{1}{2})^{1/2}$ or $(\frac{1}{2})^{1/3}$
 (c) $7^{1/4}$ or $4^{1/3}$ (d) $\sqrt[3]{5}$ or $\sqrt{3}$

1.3 ALGEBRAIC EXPRESSIONS

Adding and Subtracting Polynomials ► Multiplying Algebraic Expressions ►
 Special Product Formulas ► Factoring Common Factors ► Factoring
 Trinomials ► Special Factoring Formulas ► Factoring by Grouping Terms

A **variable** is a letter that can represent any number from a given set of numbers. If we start with variables, such as x , y , and z and some real numbers, and combine them using addition, subtraction, multiplication, division, powers, and roots, we obtain an **algebraic expression**. Here are some examples:

$$2x^2 - 3x + 4 \qquad \sqrt{x} + 10 \qquad \frac{y - 2z}{y^2 + 4}$$

A **monomial** is an expression of the form ax^k , where a is a real number and k is a non-negative integer. A **binomial** is a sum of two monomials and a **trinomial** is a sum of three monomials. In general, a sum of monomials is called a *polynomial*. For example, the first expression listed above is a polynomial, but the other two are not.

POLYNOMIALS

A **polynomial** in the variable x is an expression of the form

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

where $a_0, a_1, \dots, a_n$ are real numbers, and n is a nonnegative integer. If $a_n \neq 0$, then the polynomial has **degree n** . The monomials $a_k x^k$ that make up the polynomial are called the **terms** of the polynomial.

Note that the degree of a polynomial is the highest power of the variable that appears in the polynomial.

Polynomial	Type	Terms	Degree
$2x^2 - 3x + 4$	trinomial	$2x^2, -3x, 4$	2
$x^8 + 5x$	binomial	$x^8, 5x$	8
$3 - x + x^2 - \frac{1}{2}x^3$	four terms	$-\frac{1}{2}x^3, x^2, -x, 3$	3
$5x + 1$	binomial	$5x, 1$	1
$9x^5$	monomial	$9x^5$	5
6	monomial	6	0

▼ Adding and Subtracting Polynomials

We **add** and **subtract** polynomials using the properties of real numbers that were discussed in Section 1.1. The idea is to combine **like terms** (that is, terms with the same variables raised to the same powers) using the Distributive Property. For instance,

Distributive Property

$ac + bc = (a + b)c$

$5x^7 + 3x^7 = (5 + 3)x^7 = 8x^7$



In subtracting polynomials, we have to remember that **if a minus sign precedes an expression in parentheses, then the sign of every term within the parentheses is changed when we remove the parentheses:**

$-(b + c) = -b - c$

[This is simply a case of the Distributive Property, $a(b + c) = ab + ac$, with $a = -1$.]

EXAMPLE 1 | Adding and Subtracting Polynomials

- (a) Find the sum $(x^3 - 6x^2 + 2x + 4) + (x^3 + 5x^2 - 7x)$.
- (b) Find the difference $(x^3 - 6x^2 + 2x + 4) - (x^3 + 5x^2 - 7x)$.

SOLUTION

(a) $(x^3 - 6x^2 + 2x + 4) + (x^3 + 5x^2 - 7x)$
 $= (x^3 + x^3) + (-6x^2 + 5x^2) + (2x - 7x) + 4$ Group like terms
 $= 2x^3 - x^2 - 5x + 4$ Combine like terms

(b) $(x^3 - 6x^2 + 2x + 4) - (x^3 + 5x^2 - 7x)$
 $= x^3 - 6x^2 + 2x + 4 - x^3 - 5x^2 + 7x$ Distributive Property
 $= (x^3 - x^3) + (-6x^2 - 5x^2) + (2x + 7x) + 4$ Group like terms
 $= -11x^2 + 9x + 4$ Combine like terms

 NOW TRY EXERCISES 15 AND 17

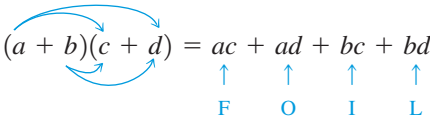
▼ Multiplying Algebraic Expressions

To find the **product** of polynomials or other algebraic expressions, we need to use the Distributive Property repeatedly. In particular, using it three times on the product of two binomials, we get

$(a + b)(c + d) = a(c + d) + b(c + d) = ac + ad + bc + bd$

This says that we multiply the two factors by multiplying each term in one factor by each term in the other factor and adding these products. Schematically, we have

The acronym **FOIL** helps us remember that the product of two binomials is the sum of the products of the **F**irst terms, the **O**uter terms, the **I**nner terms, and the **L**ast terms.

 $(a + b)(c + d) = ac + ad + bc + bd$
F O I L

In general, we can multiply two algebraic expressions by using the Distributive Property and the Laws of Exponents.

EXAMPLE 2 | Multiplying Binomials Using FOIL

$$\begin{aligned}
 (2x + 1)(3x - 5) &= 6x^2 - 10x + 3x - 5 && \text{Distributive Property} \\
 &= 6x^2 - 7x - 5 && \text{Combine like terms}
 \end{aligned}$$

$\begin{matrix} \uparrow & \uparrow & \uparrow & \uparrow \\ \text{F} & \text{O} & \text{I} & \text{L} \end{matrix}$

NOW TRY EXERCISE 23

When we multiply trinomials or other polynomials with more terms, we use the Distributive Property. It is also helpful to arrange our work in table form. The next example illustrates both methods.

EXAMPLE 3 | Multiplying Polynomials

Find the product: $(2x + 3)(x^2 - 5x + 4)$

SOLUTION 1: Using the Distributive Property

$$\begin{aligned}
 (2x + 3)(x^2 - 5x + 4) &= 2x(x^2 - 5x + 4) + 3(x^2 - 5x + 4) && \text{Distributive Property} \\
 &= (2x \cdot x^2 - 2x \cdot 5x + 2x \cdot 4) + (3 \cdot x^2 - 3 \cdot 5x + 3 \cdot 4) && \text{Distributive Property} \\
 &= (2x^3 - 10x^2 + 8x) + (3x^2 - 15x + 12) && \text{Laws of Exponents} \\
 &= 2x^3 - 7x^2 - 7x + 12 && \text{Combine like terms}
 \end{aligned}$$

SOLUTION 2: Using Table Form

$x^2 - 5x + 4$	
$\underline{2x + 3}$	
$3x^2 - 15x + 12$	Multiply $x^2 - 5x + 4$ by 3
$\underline{2x^3 - 10x^2 + 8x}$	Multiply $x^2 - 5x + 4$ by $2x$
$2x^3 - 7x^2 - 7x + 12$	Add like terms

NOW TRY EXERCISE 45

▼ Special Product Formulas

Certain types of products occur so frequently that you should memorize them. You can verify the following formulas by performing the multiplications.

See the *Discovery Project* referenced on page 34 for a geometric interpretation of some of these formulas.

SPECIAL PRODUCT FORMULAS

If A and B are any real numbers or algebraic expressions, then

1. $(A + B)(A - B) = A^2 - B^2$ Sum and product of same terms
2. $(A + B)^2 = A^2 + 2AB + B^2$ Square of a sum
3. $(A - B)^2 = A^2 - 2AB + B^2$ Square of a difference
4. $(A + B)^3 = A^3 + 3A^2B + 3AB^2 + B^3$ Cube of a sum
5. $(A - B)^3 = A^3 - 3A^2B + 3AB^2 - B^3$ Cube of a difference

The key idea in using these formulas (or any other formula in algebra) is the **Principle of Substitution**: We may substitute any algebraic expression for any letter in a formula. For example, to find $(x^2 + y^3)^2$ we use Product Formula 2, substituting x^2 for A and y^3 for B , to get

$$(x^2 + y^3)^2 = (x^2)^2 + 2(x^2)(y^3) + (y^3)^2$$

$$(A + B)^2 = A^2 + 2AB + B^2$$

EXAMPLE 4 | Using the Special Product Formulas

Use the Special Product Formulas to find each product.

(a) $(3x + 5)^2$ (b) $(x^2 - 2)^3$

SOLUTION

(a) Substituting $A = 3x$ and $B = 5$ in Product Formula 2, we get

$$(3x + 5)^2 = (3x)^2 + 2(3x)(5) + 5^2 = 9x^2 + 30x + 25$$

(b) Substituting $A = x^2$ and $B = 2$ in Product Formula 5, we get

$$\begin{aligned}(x^2 - 2)^3 &= (x^2)^3 - 3(x^2)^2(2) + 3(x^2)(2)^2 - 2^3 \\ &= x^6 - 6x^4 + 12x^2 - 8\end{aligned}$$

 NOW TRY EXERCISES 29 AND 41

EXAMPLE 5 | Using the Special Product Formulas

Find each product.

(a) $(2x - \sqrt{y})(2x + \sqrt{y})$ (b) $(x + y - 1)(x + y + 1)$

SOLUTION

(a) Substituting $A = 2x$ and $B = \sqrt{y}$ in Product Formula 1, we get

$$(2x - \sqrt{y})(2x + \sqrt{y}) = (2x)^2 - (\sqrt{y})^2 = 4x^2 - y$$

(b) If we group $x + y$ together and think of this as one algebraic expression, we can use Product Formula 1 with $A = x + y$ and $B = 1$.

$$\begin{aligned}(x + y - 1)(x + y + 1) &= [(x + y) - 1][(x + y) + 1] \\ &= (x + y)^2 - 1^2 && \text{Product Formula 1} \\ &= x^2 + 2xy + y^2 - 1 && \text{Product Formula 2}\end{aligned}$$

 NOW TRY EXERCISES 55 AND 59

▼ Factoring Common Factors

We use the Distributive Property to expand algebraic expressions. We sometimes need to reverse this process (again using the Distributive Property) by **factoring** an expression as a product of simpler ones. For example, we can write

$$\begin{array}{c} \text{FACTORED} \rightarrow \\ x^2 - 4 = (x - 2)(x + 2) \\ \leftarrow \text{EXPANDED} \end{array}$$

We say that $x - 2$ and $x + 2$ are **factors** of $x^2 - 4$.

The easiest type of factoring occurs when the terms have a common factor.

EXAMPLE 6 | Factoring Out Common Factors

Factor each expression.

- (a) $3x^2 - 6x$ (b) $8x^4y^2 + 6x^3y^3 - 2xy^4$
 (c) $(2x + 4)(x - 3) - 5(x - 3)$

CHECK YOUR ANSWER

Multiplying gives

$$3x(x - 2) = 3x^2 - 6x \quad \checkmark$$

CHECK YOUR ANSWER

Multiplying gives

$$\begin{aligned} 2xy^2(4x^3 + 3x^2y - y^2) \\ = 8x^4y^2 + 6x^3y^3 - 2xy^4 \quad \checkmark \end{aligned}$$

SOLUTION

- (a) The greatest common factor of the terms $3x^2$ and $-6x$ is $3x$, so we have

$$3x^2 - 6x = 3x(x - 2)$$

- (b) We note that

8, 6, and -2 have the greatest common factor 2

x^4 , x^3 , and x have the greatest common factor x

y^2 , y^3 , and y^4 have the greatest common factor y^2

So the greatest common factor of the three terms in the polynomial is $2xy^2$, and we have

$$\begin{aligned} 8x^4y^2 + 6x^3y^3 - 2xy^4 &= (2xy^2)(4x^3) + (2xy^2)(3x^2y) + (2xy^2)(-y^2) \\ &= 2xy^2(4x^3 + 3x^2y - y^2) \end{aligned}$$

- (c) The two terms have the common factor $x - 3$.

$$\begin{aligned} (2x + 4)(x - 3) - 5(x - 3) &= [(2x + 4) - 5](x - 3) && \text{Distributive Property} \\ &= (2x - 1)(x - 3) && \text{Simplify} \end{aligned}$$

 NOW TRY EXERCISES 61, 63, AND 65

Factoring Trinomials

To factor a trinomial of the form $x^2 + bx + c$, we note that

$$(x + r)(x + s) = x^2 + (r + s)x + rs$$

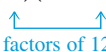
so we need to choose numbers r and s so that $r + s = b$ and $rs = c$.

EXAMPLE 7 | Factoring $x^2 + bx + c$ by Trial and Error

Factor: $x^2 + 7x + 12$

SOLUTION We need to find two integers whose product is 12 and whose sum is 7. By trial and error we find that the two integers are 3 and 4. Thus, the factorization is

$$x^2 + 7x + 12 = (x + 3)(x + 4)$$



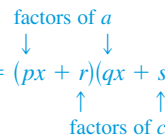
 NOW TRY EXERCISE 67

To factor a trinomial of the form $ax^2 + bx + c$ with $a \neq 1$, we look for factors of the form $px + r$ and $qx + s$:

$$ax^2 + bx + c = (px + r)(qx + s) = pqx^2 + (ps + qr)x + rs$$

Therefore, we try to find numbers p , q , r , and s such that $pq = a$, $rs = c$, $ps + qr = b$. If these numbers are all integers, then we will have a limited number of possibilities to try for p , q , r , and s .

$$ax^2 + bx + c = (px + r)(qx + s)$$



EXAMPLE 8 | Factoring $ax^2 + bx + c$ by Trial and ErrorFactor: $6x^2 + 7x - 5$ **SOLUTION** We can factor 6 as $6 \cdot 1$ or $3 \cdot 2$, and -5 as $-5 \cdot 1$ or $5 \cdot (-1)$. By trying these possibilities, we arrive at the factorization**CHECK YOUR ANSWER**

Multiplying gives

$$(3x + 5)(2x - 1) = 6x^2 + 7x - 5 \quad \checkmark$$

$$6x^2 + 7x - 5 = (3x + 5)(2x - 1)$$

$\xrightarrow{\text{factors of } 6}$
 $\xrightarrow{\text{factors of } -5}$

 **NOW TRY EXERCISE 69****EXAMPLE 9** | Recognizing the Form of an Expression

Factor each expression.

(a) $x^2 - 2x - 3$ (b) $(5a + 1)^2 - 2(5a + 1) - 3$

SOLUTION

(a) $x^2 - 2x - 3 = (x - 3)(x + 1)$ Trial and error

(b) This expression is of the form

$$\square^2 - 2\square - 3$$

where $\square$ represents $5a + 1$. This is the same form as the expression in part (a), so it will factor as $(\square - 3)(\square + 1)$.

$$\begin{aligned}
 (5a + 1)^2 - 2(5a + 1) - 3 &= [(5a + 1) - 3][(5a + 1) + 1] \\
 &= (5a - 2)(5a + 2)
 \end{aligned}$$

 **NOW TRY EXERCISE 71****Special Factoring Formulas**

Some special algebraic expressions can be factored using the following formulas. The first three are simply Special Product Formulas written backward.

SPECIAL FACTORING FORMULAS

Formula	Name
1. $A^2 - B^2 = (A - B)(A + B)$	Difference of squares
2. $A^2 + 2AB + B^2 = (A + B)^2$	Perfect square
3. $A^2 - 2AB + B^2 = (A - B)^2$	Perfect square
4. $A^3 - B^3 = (A - B)(A^2 + AB + B^2)$	Difference of cubes
5. $A^3 + B^3 = (A + B)(A^2 - AB + B^2)$	Sum of cubes

EXAMPLE 10 | Factoring Differences of Squares

Factor each expression.

(a) $4x^2 - 25$ (b) $(x + y)^2 - z^2$

MATHEMATICS IN THE MODERN WORLD

Changing Words, Sound, and Pictures into Numbers

Pictures, sound, and text are routinely transmitted from one place to another via the Internet, fax machines, or modems. How can such things be transmitted through telephone wires? The key to doing this is to change them into numbers or bits (the digits 0 or 1). It's easy to see how to change text to numbers. For example, we could use the correspondence $A = 00000001$, $B = 00000010$, $C = 00000011$, $D = 00000100$, $E = 00000101$, and so on. The word "BED" then becomes 0000001000000010100000100. By reading the digits in groups of eight, it is possible to translate this number back to the word "BED."

Changing sound to bits is more complicated. A sound wave can be graphed on an oscilloscope or a computer. The graph is then broken down mathematically into simpler components corresponding to the different frequencies of the original sound. (A branch of mathematics called Fourier analysis is used here.) The intensity of each component is a number, and the original sound can be reconstructed from these numbers. For example, music is stored on a CD as a sequence of bits; it may look like 101010001010010100101010 10000010 11110101000101011.... (One second of music requires 1.5 million bits!) The CD player reconstructs the music from the numbers on the CD.

Changing pictures into numbers involves expressing the color and brightness of each dot (or pixel) into a number. This is done very efficiently using a branch of mathematics called wavelet theory. The FBI uses wavelets as a compact way to store the millions of fingerprints they need on file.

SOLUTION

- (a) Using the Difference of Squares Formula with $A = 2x$ and $B = 5$, we have

$$4x^2 - 25 = (2x)^2 - 5^2 = (2x - 5)(2x + 5)$$

$$A^2 - B^2 = (A - B)(A + B)$$

- (b) We use the Difference of Squares Formula with $A = x + y$ and $B = z$.

$$(x + y)^2 - z^2 = (x + y - z)(x + y + z)$$

NOW TRY EXERCISES 75 AND 109

EXAMPLE 11 | Factoring Differences and Sums of Cubes

Factor each polynomial.

- (a) $27x^3 - 1$ (b) $x^6 + 8$

SOLUTION

- (a) Using the Difference of Cubes Formula with $A = 3x$ and $B = 1$, we get

$$\begin{aligned} 27x^3 - 1 &= (3x)^3 - 1^3 = (3x - 1)[(3x)^2 + (3x)(1) + 1^2] \\ &= (3x - 1)(9x^2 + 3x + 1) \end{aligned}$$

- (b) Using the Sum of Cubes Formula with $A = x^2$ and $B = 2$, we have

$$x^6 + 8 = (x^2)^3 + 2^3 = (x^2 + 2)(x^4 - 2x^2 + 4)$$

NOW TRY EXERCISES 77 AND 79

A trinomial is a perfect square if it is of the form

$$A^2 + 2AB + B^2 \quad \text{or} \quad A^2 - 2AB + B^2$$

So we **recognize a perfect square** if the middle term ($2AB$ or $-2AB$) is plus or minus twice the product of the square roots of the outer two terms.

EXAMPLE 12 | Recognizing Perfect Squares

Factor each trinomial.

- (a) $x^2 + 6x + 9$ (b) $4x^2 - 4xy + y^2$

SOLUTION

- (a) Here $A = x$ and $B = 3$, so $2AB = 2 \cdot x \cdot 3 = 6x$. Since the middle term is $6x$, the trinomial is a perfect square. By the Perfect Square Formula we have

$$x^2 + 6x + 9 = (x + 3)^2$$

- (b) Here $A = 2x$ and $B = y$, so $2AB = 2 \cdot 2x \cdot y = 4xy$. Since the middle term is $-4xy$, the trinomial is a perfect square. By the Perfect Square Formula we have

$$4x^2 - 4xy + y^2 = (2x - y)^2$$

NOW TRY EXERCISES 105 AND 107

When we factor an expression, the result can sometimes be factored further. In general, we *first factor out common factors*, then inspect the result to see whether it can be factored by any of the other methods of this section. We repeat this process until we have factored the expression completely.

EXAMPLE 13 | Factoring an Expression Completely

Factor each expression completely.

(a) $2x^4 - 8x^2$ (b) $x^5y^2 - xy^6$

SOLUTION(a) We first factor out the power of x with the smallest exponent.

$$\begin{aligned} 2x^4 - 8x^2 &= 2x^2(x^2 - 4) && \text{Common factor is } 2x^2 \\ &= 2x^2(x - 2)(x + 2) && \text{Factor } x^2 - 4 \text{ as a difference of squares} \end{aligned}$$

(b) We first factor out the powers of x and y with the smallest exponents.

$$\begin{aligned} x^5y^2 - xy^6 &= xy^2(x^4 - y^4) && \text{Common factor is } xy^2 \\ &= xy^2(x^2 + y^2)(x^2 - y^2) && \text{Factor } x^4 - y^4 \text{ as a difference of squares} \\ &= xy^2(x^2 + y^2)(x + y)(x - y) && \text{Factor } x^2 - y^2 \text{ as a difference of squares} \end{aligned}$$

 NOW TRY EXERCISES 115 AND 117

In the next example we factor out variables with fractional exponents. This type of factoring occurs in calculus.

EXAMPLE 14 | Factoring Expressions with Fractional Exponents

Factor each expression.

(a) $3x^{3/2} - 9x^{1/2} + 6x^{-1/2}$ (b) $(2 + x)^{-2/3}x + (2 + x)^{1/3}$

SOLUTION(a) Factor out the power of x with the *smallest exponent*, that is, $x^{-1/2}$.

$$\begin{aligned} 3x^{3/2} - 9x^{1/2} + 6x^{-1/2} &= 3x^{-1/2}(x^2 - 3x + 2) && \text{Factor out } 3x^{-1/2} \\ &= 3x^{-1/2}(x - 1)(x - 2) && \text{Factor the quadratic } x^2 - 3x + 2 \end{aligned}$$

(b) Factor out the power of $2 + x$ with the *smallest exponent*, that is, $(2 + x)^{-2/3}$.

$$\begin{aligned} (2 + x)^{-2/3}x + (2 + x)^{1/3} &= (2 + x)^{-2/3}[x + (2 + x)] && \text{Factor out } (2 + x)^{-2/3} \\ &= (2 + x)^{-2/3}(2 + 2x) && \text{Simplify} \\ &= 2(2 + x)^{-2/3}(1 + x) && \text{Factor out 2} \end{aligned}$$

To factor out $x^{-1/2}$ from $x^{3/2}$, we subtract exponents:

$$\begin{aligned} x^{3/2} &= x^{-1/2}(x^{3/2 - (-1/2)}) \\ &= x^{-1/2}(x^{3/2 + 1/2}) \\ &= x^{-1/2}(x^2) \end{aligned}$$

CHECK YOUR ANSWERS

To see that you have factored correctly, multiply using the Laws of Exponents.

$$\begin{aligned} \text{(a)} \quad 3x^{-1/2}(x^2 - 3x + 2) &= 3x^{3/2} - 9x^{1/2} + 6x^{-1/2} && \checkmark \\ \text{(b)} \quad (2 + x)^{-2/3}[x + (2 + x)] &= (2 + x)^{-2/3}x + (2 + x)^{1/3} && \checkmark \end{aligned}$$

 NOW TRY EXERCISES 91 AND 93**Factoring by Grouping Terms**

Polynomials with at least four terms can sometimes be factored by grouping terms. The following example illustrates the idea.

EXAMPLE 15 | Factoring by Grouping

Factor each polynomial.

(a) $x^3 + x^2 + 4x + 4$ (b) $x^3 - 2x^2 - 3x + 6$

SOLUTION

$$(a) \ x^3 + x^2 + 4x + 4 = (x^3 + x^2) + (4x + 4)$$

$$= x^2(x + 1) + 4(x + 1)$$

$$= (x^2 + 4)(x + 1)$$

$$(b) \ x^3 - 2x^2 - 3x + 6 = (x^3 - 2x^2) - (3x - 6)$$

$$= x^2(x - 2) - 3(x - 2)$$

$$= (x^2 - 3)(x - 2)$$

Group terms

Factor out common factors

Factor out $x + 1$ from each term

Group terms

Factor out common factors

Factor out $x - 2$ from each term **NOW TRY EXERCISE 83****1.3 EXERCISES****CONCEPTS**

1. Consider the polynomial
- $2x^5 + 6x^4 + 4x^3$
- .

How many terms does this polynomial have? _____

List the terms: _____

What factor is common to each term? _____

Factor the polynomial: $2x^5 + 6x^4 + 4x^3 =$ _____.

2. To factor the trinomial
- $x^2 + 7x + 10$
- , we look for two integers whose product is _____ and whose sum is _____.

These integers are _____ and _____, so the trinomial factors as _____.

3. The Special Product Formula for the “square of a sum” is

$$(A + B)^2 = \text{_____}.$$

$$\text{So } (2x + 3)^2 = \text{_____}.$$

4. The Special Product Formula for the “sum and difference of the same terms” is
- $(A + B)(A - B) =$
- _____.

$$\text{So } (5 + x)(5 - x) = \text{_____}.$$

5. The Special Factoring Formula for the “difference of squares” is

$$A^2 - B^2 = \text{_____}. \text{ So } 4x^2 - 25 \text{ factors as } \text{_____}.$$

6. The Special Factoring Formula for a “perfect square” is

$$A^2 + 2AB + B^2 = \text{_____}. \text{ So } x^2 + 10x + 25 \text{ factors as } \text{_____}.$$

Polynomial	Type	Terms	Degree
9. -8			
10. $\frac{1}{2}x^7$			
11. $x - x^2 + x^3 - x^4$			
12. $\sqrt{2}x - \sqrt{3}$			

- 13–22 ■ Find the sum, difference, or product.

13. $(12x - 7) - (5x - 12)$ 14. $(5 - 3x) + (2x - 8)$

 15. $(3x^2 + x + 1) + (2x^2 - 3x - 5)$

16. $(3x^2 + x + 1) - (2x^2 - 3x - 5)$

 17. $(x^3 + 6x^2 - 4x + 7) - (3x^2 + 2x - 4)$

18. $3(x - 1) + 4(x + 2)$

19. $8(2x + 5) - 7(x - 9)$

20. $4(x^2 - 3x + 5) - 3(x^2 - 2x + 1)$

21. $2(2 - 5t) + t^2(t - 1) - (t^4 - 1)$

22. $5(3t - 4) - (t^2 + 2) - 2t(t - 3)$

- 23–28 ■ Multiply the algebraic expressions using the FOIL method and simplify.

 23. $(3t - 2)(7t - 4)$

24. $(4s - 1)(2s + 5)$

25. $(3x + 5)(2x - 1)$

26. $(7y - 3)(2y - 1)$

27. $(x + 3y)(2x - y)$

28. $(4x - 5y)(3x - y)$

- 29–44 ■ Multiply the algebraic expressions using a Special Product Formula and simplify.

 29. $(3x + 4)^2$

30. $(1 - 2y)^2$

31. $(2u + v)^2$

32. $(x - 3y)^2$

33. $(2x + 3y)^2$

34. $(r - 2s)^2$

35. $(x + 5)(x - 5)$

36. $(y - 3)(y + 3)$

37. $(3x - 4)(3x + 4)$

38. $(2y + 5)(2y - 5)$

39. $(\sqrt{x} + 2)(\sqrt{x} - 2)$

40. $(\sqrt{y} + \sqrt{2})(\sqrt{y} - \sqrt{2})$

SKILLS

- 7–12 ■ Complete the following table by stating whether the polynomial is a monomial, binomial, or trinomial; then list its terms and state its degree.

Polynomial	Type	Terms	Degree
7. $x^2 - 3x + 7$			
8. $2x^5 + 4x^2$			

41. $(y + 2)^3$ 42. $(x - 3)^3$
 43. $(1 - 2r)^3$ 44. $(3 + 2y)^3$
- 45–60** ■ Perform the indicated operations and simplify.
45. $(x + 2)(x^2 + 2x + 3)$ 46. $(x + 1)(2x^2 - x + 1)$
 47. $(2x - 5)(x^2 - x + 1)$ 48. $(1 + 2x)(x^2 - 3x + 1)$
 49. $\sqrt{x}(x - \sqrt{x})$ 50. $x^{3/2}(\sqrt{x} - 1/\sqrt{x})$
 51. $y^{1/3}(y^{2/3} + y^{5/3})$ 52. $x^{1/4}(2x^{3/4} - x^{1/4})$
 53. $(x^2 - a^2)(x^2 + a^2)$ 54. $(x^{1/2} + y^{1/2})(x^{1/2} - y^{1/2})$
55. $(\sqrt{a} - b)(\sqrt{a} + b)$
 56. $(\sqrt{h^2 + 1} + 1)(\sqrt{h^2 + 1} - 1)$
 57. $((x - 1) + x^2)((x - 1) - x^2)$
 58. $(x + (2 + x^2))(x - (2 + x^2))$
59. $(2x + y - 3)(2x + y + 3)$ 60. $(x + y + z)(x - y - z)$

61–66 ■ Factor out the common factor.

61. $-2x^3 + 16x$ 62. $2x^4 + 4x^3 - 14x^2$
 63. $y(y - 6) + 9(y - 6)$ 64. $(z + 2)^2 - 5(z + 2)$
 65. $2x^2y - 6xy^2 + 3xy$ 66. $-7x^4y^2 + 14xy^3 + 21xy^4$

67–74 ■ Factor the trinomial.

67. $x^2 + 2x - 3$ 68. $x^2 - 6x + 5$
 69. $8x^2 - 14x - 15$ 70. $6y^2 + 11y - 21$
 71. $3x^2 - 16x + 5$ 72. $5x^2 - 7x - 6$
 73. $(3x + 2)^2 + 8(3x + 2) + 12$
 74. $2(a + b)^2 + 5(a + b) - 3$

75–82 ■ Use a Special Factoring Formula to factor the expression.

75. $9a^2 - 16$ 76. $(x + 3)^2 - 4$
 77. $27x^3 + y^3$ 78. $a^3 - b^6$
 79. $8s^3 - 125t^3$ 80. $1 + 1000y^3$
 81. $x^2 + 12x + 36$ 82. $16z^2 - 24z + 9$

83–88 ■ Factor the expression by grouping terms.

83. $x^3 + 4x^2 + x + 4$ 84. $3x^3 - x^2 + 6x - 2$
 85. $2x^3 + x^2 - 6x - 3$ 86. $-9x^3 - 3x^2 + 3x + 1$
 87. $x^3 + x^2 + x + 1$ 88. $x^5 + x^4 + x + 1$

89–94 ■ Factor the expression completely. Begin by factoring out the lowest power of each common factor.

89. $x^{5/2} - x^{1/2}$ 90. $3x^{-1/2} + 4x^{1/2} + x^{3/2}$
 91. $x^{-3/2} + 2x^{-1/2} + x^{1/2}$ 92. $(x - 1)^{7/2} - (x - 1)^{3/2}$
 93. $(x^2 + 1)^{1/2} + 2(x^2 + 1)^{-1/2}$
 94. $x^{-1/2}(x + 1)^{1/2} + x^{1/2}(x + 1)^{-1/2}$

95–124 ■ Factor the expression completely.

95. $12x^3 + 18x$ 96. $30x^3 + 15x^4$
 97. $x^2 - 2x - 8$ 98. $x^2 - 14x + 48$

99. $2x^2 + 5x + 3$ 100. $2x^2 + 7x - 4$
 101. $9x^2 - 36x - 45$ 102. $8x^2 + 10x + 3$
 103. $49 - 4y^2$ 104. $4t^2 - 9s^2$
 105. $t^2 - 6t + 9$ 106. $x^2 + 10x + 25$
 107. $4x^2 + 4xy + y^2$ 108. $r^2 - 6rs + 9s^2$
 109. $(a + b)^2 - (a - b)^2$ 110. $\left(1 + \frac{1}{x}\right)^2 - \left(1 - \frac{1}{x}\right)^2$
 111. $x^2(x^2 - 1) - 9(x^2 - 1)$ 112. $(a^2 - 1)b^2 - 4(a^2 - 1)$
 113. $8x^3 - 125$ 114. $x^6 + 64$
 115. $x^3 + 2x^2 + x$ 116. $3x^3 - 27x$
 117. $x^4y^3 - x^2y^5$ 118. $18y^3x^2 - 2xy^4$
 119. $2x^3 + 4x^2 + x + 2$ 120. $3x^3 + 5x^2 - 6x - 10$
 121. $(x - 1)(x + 2)^2 - (x - 1)^2(x + 2)$
 122. $y^4(y + 2)^3 + y^5(y + 2)^4$
 123. $(a^2 + 1)^2 - 7(a^2 + 1) + 10$
 124. $(a^2 + 2a)^2 - 2(a^2 + 2a) - 3$

125–128 ■ Factor the expression completely. (This type of expression arises in calculus when using the “Product Rule.”)

125. $5(x^2 + 4)^4(2x)(x - 2)^4 + (x^2 + 4)^5(4)(x - 2)^3$
 126. $3(2x - 1)^2(2)(x + 3)^{1/2} + (2x - 1)^3(\frac{1}{2})(x + 3)^{-1/2}$
 127. $(x^2 + 3)^{-1/3} - \frac{2}{3}x^2(x^2 + 3)^{-4/3}$
 128. $\frac{1}{2}x^{-1/2}(3x + 4)^{1/2} - \frac{3}{2}x^{1/2}(3x + 4)^{-1/2}$
 129. (a) Show that $ab = \frac{1}{2}[(a + b)^2 - (a^2 + b^2)]$.
 (b) Show that $(a^2 + b^2)^2 - (a^2 - b^2)^2 = 4a^2b^2$.
 (c) Show that
 $(a^2 + b^2)(c^2 + d^2) = (ac + bd)^2 + (ad - bc)^2$
 (d) Factor completely: $4a^2c^2 - (a^2 - b^2 + c^2)^2$.

130. Verify Special Factoring Formulas 4 and 5 by expanding their right-hand sides.

APPLICATIONS

- 131. Volume of Concrete** A culvert is constructed out of large cylindrical shells cast in concrete, as shown in the figure. Using the formula for the volume of a cylinder given on the inside front cover of this book, explain why the volume of the cylindrical shell is

$$V = \pi R^2 h - \pi r^2 h$$

Factor to show that

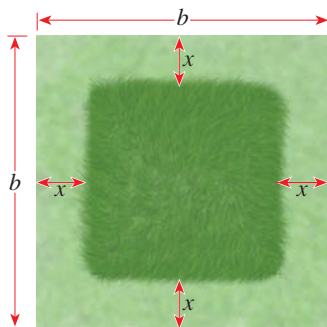
$$V = 2\pi \cdot \text{average radius} \cdot \text{height} \cdot \text{thickness}$$

Use the “unrolled” diagram to explain why this makes sense geometrically.



- 132. Mowing a Field** A square field in a certain state park is mowed around the edges every week. The rest of the field is kept unmowed to serve as a habitat for birds and small animals (see the figure). The field measures b feet by b feet, and the mowed strip is x feet wide.

- (a) Explain why the area of the mowed portion is $b^2 - (b - 2x)^2$.
 (b) Factor the expression in part (a) to show that the area of the mowed portion is also $4x(b - x)$.



DISCOVERY ■ DISCUSSION ■ WRITING

133. Degrees of Sums and Products of Polynomials

Make up several pairs of polynomials, then calculate the sum and product of each pair. On the basis of your experiments and observations, answer the following questions.

- (a) How is the degree of the product related to the degrees of the original polynomials?
 (b) How is the degree of the sum related to the degrees of the original polynomials?

- 134. The Power of Algebraic Formulas** Use the Difference of Squares Formula to factor $17^2 - 16^2$. Notice that it is easy to calculate the factored form in your head but not so easy to calculate the original form in this way. Evaluate each expression in your head:

- (a) $528^2 - 527^2$
 (b) $122^2 - 120^2$
 (c) $1020^2 - 1010^2$

Now use the Special Product Formula

$$(A + B)(A - B) = A^2 - B^2$$

to evaluate these products in your head:

- (d) $79 \cdot 51$
 (e) $998 \cdot 1002$

135. Differences of Even Powers

- (a) Factor the expressions completely: $A^4 - B^4$ and $A^6 - B^6$.
 (b) Verify that $18,335 = 12^4 - 7^4$ and that $2,868,335 = 12^6 - 7^6$.
 (c) Use the results of parts (a) and (b) to factor the integers 18,335 and 2,868,335. Then show that in both of these factorizations, all the factors are prime numbers.

- 136. Factoring $A^n - 1$** Verify these formulas by expanding and simplifying the right-hand side.

$$A^2 - 1 = (A - 1)(A + 1)$$

$$A^3 - 1 = (A - 1)(A^2 + A + 1)$$

$$A^4 - 1 = (A - 1)(A^3 + A^2 + A + 1)$$

On the basis of the pattern displayed in this list, how do you think $A^5 - 1$ would factor? Verify your conjecture. Now generalize the pattern you have observed to obtain a factoring formula for $A^n - 1$, where n is a positive integer.

- 137. Factoring $x^4 + ax^2 + b$** A trinomial of the form $x^4 + ax^2 + b$ can sometimes be factored easily. For example,

$$x^4 + 3x^2 - 4 = (x^2 + 4)(x^2 - 1)$$

But $x^4 + 3x^2 + 4$ cannot be factored in this way. Instead, we can use the following method.

$$x^4 + 3x^2 + 4 = (x^4 + 4x^2 + 4) - x^2$$

Add and subtract x^2

$$= (x^2 + 2)^2 - x^2$$

Factor perfect square

$$= [(x^2 + 2) - x][(x^2 + 2) + x]$$

Difference of squares

$$= (x^2 - x + 2)(x^2 + x + 2)$$

Factor the following, using whichever method is appropriate.

- (a) $x^4 + x^2 - 2$
 (b) $x^4 + 2x^2 + 9$
 (c) $x^4 + 4x^2 + 16$
 (d) $x^4 + 2x^2 + 1$



DISCOVERY PROJECT

Visualizing a Formula

In this project we discover geometric interpretations of some of the Special Product Formulas. You can find the project at the book companion website: www.stewartmath.com

1.4 RATIONAL EXPRESSIONS

The Domain of an Algebraic Expression ► Simplifying Rational Expressions
 ► Multiplying and Dividing Rational Expressions ► Adding and Subtracting
 Rational Expressions ► Compound Fractions ► Rationalizing the
 Denominator or the Numerator ► Avoiding Common Errors

A quotient of two algebraic expressions is called a **fractional expression**. Here are some examples:

$$\frac{2x}{x-1} \quad \frac{\sqrt{x}+3}{x+1} \quad \frac{y-2}{y^2+4}$$

A **rational expression** is a fractional expression where both the numerator and denominator are polynomials. For example, the following are rational expressions:

$$\frac{2x}{x-1} \quad \frac{x}{x^2+1} \quad \frac{x^3-x}{x^2-5x+6}$$

In this section we learn how to perform algebraic operations on rational expressions.

▼ The Domain of an Algebraic Expression

Expression	Domain
$\frac{1}{x}$	$\{x \mid x \neq 0\}$
$\sqrt{x}$	$\{x \mid x \geq 0\}$
$\frac{1}{\sqrt{x}}$	$\{x \mid x > 0\}$

In general, an algebraic expression may not be defined for all values of the variable. The **domain** of an algebraic expression is the set of real numbers that the variable is permitted to have. The table in the margin gives some basic expressions and their domains.

EXAMPLE 1 | Finding the Domain of an Expression

Find the domains of the following expressions.

(a) $2x^2 + 3x - 1$ (b) $\frac{x}{x^2 - 5x + 6}$ (c) $\frac{\sqrt{x}}{x - 5}$

SOLUTION

(a) This polynomial is defined for every x . Thus, the domain is the set $\mathbb{R}$ of real numbers.

(b) We first factor the denominator.

$$\frac{x}{x^2 - 5x + 6} = \frac{x}{(x-2)(x-3)}$$

Denominator would be 0 if
 $x = 2$ or $x = 3$

Since the denominator is zero when $x = 2$ or 3 , the expression is not defined for these numbers. The domain is $\{x \mid x \neq 2 \text{ and } x \neq 3\}$.

(c) For the numerator to be defined, we must have $x \geq 0$. Also, we cannot divide by zero, so $x \neq 5$.

Must have $x \geq 0$
 to take square root

$$\frac{\sqrt{x}}{x-5}$$

Denominator would
 be 0 if $x = 5$

Thus, the domain is $\{x \mid x \geq 0 \text{ and } x \neq 5\}$.

✎ NOW TRY EXERCISE 11

▼ Simplifying Rational Expressions

To **simplify rational expressions**, we factor both numerator and denominator and use the following property of fractions:

$$\frac{AC}{BC} = \frac{A}{B}$$

This allows us to **cancel** common factors from the numerator and denominator.

EXAMPLE 2 | Simplifying Rational Expressions by Cancellation

Simplify: $\frac{x^2 - 1}{x^2 + x - 2}$

SOLUTION

 We can't cancel the x^2 's in $\frac{x^2 - 1}{x^2 + x - 2}$ because x^2 is not a factor.

$$\begin{aligned} \frac{x^2 - 1}{x^2 + x - 2} &= \frac{(x - 1)(x + 1)}{(x - 1)(x + 2)} && \text{Factor} \\ &= \frac{x + 1}{x + 2} && \text{Cancel common factors} \end{aligned}$$

 NOW TRY EXERCISE 17

▼ Multiplying and Dividing Rational Expressions

To **multiply rational expressions**, we use the following property of fractions:

$$\frac{A}{B} \cdot \frac{C}{D} = \frac{AC}{BD}$$

This says that to multiply two fractions we multiply their numerators and multiply their denominators.

EXAMPLE 3 | Multiplying Rational Expressions

Perform the indicated multiplication and simplify: $\frac{x^2 + 2x - 3}{x^2 + 8x + 16} \cdot \frac{3x + 12}{x - 1}$

SOLUTION We first factor.

$$\begin{aligned} \frac{x^2 + 2x - 3}{x^2 + 8x + 16} \cdot \frac{3x + 12}{x - 1} &= \frac{(x - 1)(x + 3)}{(x + 4)^2} \cdot \frac{3(x + 4)}{x - 1} && \text{Factor} \\ &= \frac{3(x - 1)(x + 3)(x + 4)}{(x - 1)(x + 4)^2} && \text{Property of fractions} \\ &= \frac{3(x + 3)}{x + 4} && \text{Cancel common factors} \end{aligned}$$

 NOW TRY EXERCISE 25

To **divide rational expressions**, we use the following property of fractions:

$$\frac{A}{B} \div \frac{C}{D} = \frac{A}{B} \cdot \frac{D}{C}$$

This says that to divide a fraction by another fraction, we invert the divisor and multiply.

EXAMPLE 4 | Dividing Rational Expressions

Perform the indicated division and simplify:

$$\frac{x-4}{x^2-4} \div \frac{x^2-3x-4}{x^2+5x+6}$$

SOLUTION

$$\begin{aligned} \frac{x-4}{x^2-4} \div \frac{x^2-3x-4}{x^2+5x+6} &= \frac{x-4}{x^2-4} \cdot \frac{x^2+5x+6}{x^2-3x-4} && \text{Invert and multiply} \\ &= \frac{(x-4)(x+2)(x+3)}{(x-2)(x+2)(x-4)(x+1)} && \text{Factor} \\ &= \frac{x+3}{(x-2)(x+1)} && \text{Cancel common factors} \end{aligned}$$

 NOW TRY EXERCISE 31

▼ Adding and Subtracting Rational Expressions

 Avoid making the following error:

$$\frac{A}{B+C} \neq \frac{A}{B} + \frac{A}{C}$$

For instance, if we let $A = 2$, $B = 1$, and $C = 1$, then we see the error:

$$\begin{aligned} \frac{2}{1+1} &\stackrel{?}{=} \frac{2}{1} + \frac{2}{1} \\ \frac{2}{2} &\stackrel{?}{=} 2 + 2 \\ 1 &\stackrel{?}{=} 4 \quad \text{Wrong!} \end{aligned}$$

To **add or subtract rational expressions**, we first find a common denominator and then use the following property of fractions:

$$\frac{A}{C} + \frac{B}{C} = \frac{A+B}{C}$$

Although any common denominator will work, it is best to use the **least common denominator** (LCD) as explained in Section 1.1. The LCD is found by factoring each denominator and taking the product of the distinct factors, using the highest power that appears in any of the factors.

EXAMPLE 5 | Adding and Subtracting Rational Expressions

Perform the indicated operations and simplify:

$$\text{(a)} \quad \frac{3}{x-1} + \frac{x}{x+2} \qquad \text{(b)} \quad \frac{1}{x^2-1} - \frac{2}{(x+1)^2}$$

SOLUTION

(a) Here the LCD is simply the product $(x-1)(x+2)$.

$$\begin{aligned} \frac{3}{x-1} + \frac{x}{x+2} &= \frac{3(x+2)}{(x-1)(x+2)} + \frac{x(x-1)}{(x-1)(x+2)} && \text{Write fractions using LCD} \\ &= \frac{3x+6+x^2-x}{(x-1)(x+2)} && \text{Add fractions} \\ &= \frac{x^2+2x+6}{(x-1)(x+2)} && \text{Combine terms in numerator} \end{aligned}$$

(b) The LCD of $x^2 - 1 = (x - 1)(x + 1)$ and $(x + 1)^2$ is $(x - 1)(x + 1)^2$.

$$\begin{aligned}
 \frac{1}{x^2 - 1} - \frac{2}{(x + 1)^2} &= \frac{1}{(x - 1)(x + 1)} - \frac{2}{(x + 1)^2} && \text{Factor} \\
 &= \frac{(x + 1) - 2(x - 1)}{(x - 1)(x + 1)^2} && \text{Combine fractions using LCD} \\
 &= \frac{x + 1 - 2x + 2}{(x - 1)(x + 1)^2} && \text{Distributive Property} \\
 &= \frac{3 - x}{(x - 1)(x + 1)^2} && \text{Combine terms in numerator}
 \end{aligned}$$

 NOW TRY EXERCISES 43 AND 45

▼ Compound Fractions

A **compound fraction** is a fraction in which the numerator, the denominator, or both, are themselves fractional expressions.

EXAMPLE 6 | Simplifying a Compound Fraction

Simplify:
$$\frac{\frac{x}{y} + 1}{1 - \frac{y}{x}}$$

SOLUTION 1 We combine the terms in the numerator into a single fraction. We do the same in the denominator. Then we invert and multiply.

$$\begin{aligned}
 \frac{\frac{x}{y} + 1}{1 - \frac{y}{x}} &= \frac{\frac{x + y}{y}}{\frac{x - y}{x}} = \frac{x + y}{y} \cdot \frac{x}{x - y} \\
 &= \frac{x(x + y)}{y(x - y)}
 \end{aligned}$$

MATHEMATICS IN THE MODERN WORLD



Courtesy of NASA

Error-Correcting Codes

The pictures sent back by the *Pathfinder* spacecraft from the surface of Mars on July 4, 1997, were astoundingly clear. But few viewing these pictures were aware of the complex mathematics used to accomplish that feat. The distance to Mars is enormous,

and the background noise (or static) is many times stronger than the original signal emitted by the spacecraft. So, when scientists receive the signal, it is full of errors. To get a clear picture, the errors must be found and corrected. This same problem of errors is routinely encountered in transmitting bank records when you use an ATM machine or voice when you are talking on the telephone.

To understand how errors are found and corrected, we must first understand that to transmit pictures, sound, or text we transform them into bits

(the digits 0 or 1; see page 30). To help the receiver recognize errors, the message is “coded” by inserting additional bits. For example, suppose you want to transmit the message “10100.” A very simple-minded code is as follows: Send each digit a million times. The person receiving the message reads it in blocks of a million digits. If the first block is mostly 1’s, he concludes that you are probably trying to transmit a 1, and so on. To say that this code is not efficient is a bit of an understatement; it requires sending a million times more data than the original message. Another method inserts “check digits.” For example, for each block of eight digits insert a ninth digit; the inserted digit is 0 if there is an even number of 1’s in the block and 1 if there is an odd number. So, if a single digit is wrong (a 0 changed to a 1, or vice versa), the check digits allow us to recognize that an error has occurred. This method does not tell us where the error is, so we can’t correct it. Modern error-correcting codes use interesting mathematical algorithms that require inserting relatively few digits but that allow the receiver to not only recognize, but also correct, errors. The first error-correcting code was developed in the 1940s by Richard Hamming at MIT. It is interesting to note that the English language has a built-in error-correcting mechanism; to test it, try reading this error-laden sentence: Gve mo libty ox giv ne deth.

SOLUTION 2 We find the LCD of all the fractions in the expression, then multiply numerator and denominator by it. In this example the LCD of all the fractions is xy . Thus

$$\begin{aligned}\frac{\frac{x}{y} + 1}{1 - \frac{y}{x}} &= \frac{\frac{x}{y} + 1}{1 - \frac{y}{x}} \cdot \frac{xy}{xy} && \text{Multiply numerator and denominator by } xy \\ &= \frac{x^2 + xy}{xy - y^2} && \text{Simplify} \\ &= \frac{x(x + y)}{y(x - y)} && \text{Factor}\end{aligned}$$

 **NOW TRY EXERCISES 59 AND 61**

The next two examples show situations in calculus that require the ability to work with fractional expressions.

EXAMPLE 7 | Simplifying a Compound Fraction

Simplify: $\frac{\frac{1}{a+h} - \frac{1}{a}}{h}$

SOLUTION We begin by combining the fractions in the numerator using a common denominator.

$$\begin{aligned}\frac{\frac{1}{a+h} - \frac{1}{a}}{h} &= \frac{a - (a+h)}{a(a+h)} && \text{Combine fractions in the numerator} \\ &= \frac{a - (a+h)}{a(a+h)} \cdot \frac{1}{h} && \text{Property 2 of fractions (invert divisor and multiply)} \\ &= \frac{a - a - h}{a(a+h)} \cdot \frac{1}{h} && \text{Distributive Property} \\ &= \frac{-h}{a(a+h)} \cdot \frac{1}{h} && \text{Simplify} \\ &= \frac{-1}{a(a+h)} && \text{Property 5 of fractions (cancel common factors)}\end{aligned}$$

 **NOW TRY EXERCISE 69**

EXAMPLE 8 | Simplifying a Compound Fraction

Simplify: $\frac{(1+x^2)^{1/2} - x^2(1+x^2)^{-1/2}}{1+x^2}$

SOLUTION 1 Factor $(1+x^2)^{-1/2}$ from the numerator.

Factor out the power of $1+x^2$ with the *smallest* exponent, in this case $(1+x^2)^{-1/2}$.

$$\begin{aligned}\frac{(1+x^2)^{1/2} - x^2(1+x^2)^{-1/2}}{1+x^2} &= \frac{(1+x^2)^{-1/2}[(1+x^2) - x^2]}{1+x^2} \\ &= \frac{(1+x^2)^{-1/2}}{1+x^2} = \frac{1}{(1+x^2)^{3/2}}\end{aligned}$$

SOLUTION 2 Since $(1 + x^2)^{-1/2} = 1/(1 + x^2)^{1/2}$ is a fraction, we can clear all fractions by multiplying numerator and denominator by $(1 + x^2)^{1/2}$.

$$\begin{aligned}\frac{(1 + x^2)^{1/2} - x^2(1 + x^2)^{-1/2}}{1 + x^2} &= \frac{(1 + x^2)^{1/2} - x^2(1 + x^2)^{-1/2}}{1 + x^2} \cdot \frac{(1 + x^2)^{1/2}}{(1 + x^2)^{1/2}} \\ &= \frac{(1 + x^2) - x^2}{(1 + x^2)^{3/2}} = \frac{1}{(1 + x^2)^{3/2}}\end{aligned}$$

 **NOW TRY EXERCISE 77**

▼ Rationalizing the Denominator or the Numerator

If a fraction has a denominator of the form $A + B\sqrt{C}$, we may rationalize the denominator by multiplying numerator and denominator by the **conjugate radical** $A - B\sqrt{C}$. This works because, by Special Product Formula 1 in Section 1.3, the product of the denominator and its conjugate radical does not contain a radical:

$$(A + B\sqrt{C})(A - B\sqrt{C}) = A^2 - B^2C$$

EXAMPLE 9 | Rationalizing the Denominator

Rationalize the denominator: $\frac{1}{1 + \sqrt{2}}$

SOLUTION We multiply both the numerator and the denominator by the conjugate radical of $1 + \sqrt{2}$, which is $1 - \sqrt{2}$.

$$\begin{aligned}\frac{1}{1 + \sqrt{2}} &= \frac{1}{1 + \sqrt{2}} \cdot \frac{1 - \sqrt{2}}{1 - \sqrt{2}} && \text{Multiply numerator and denominator by the conjugate radical} \\ &= \frac{1 - \sqrt{2}}{1^2 - (\sqrt{2})^2} && \text{Special Product Formula 1} \\ &= \frac{1 - \sqrt{2}}{1 - 2} = \frac{1 - \sqrt{2}}{-1} = \sqrt{2} - 1\end{aligned}$$

Special Product Formula 1
 $(A + B)(A - B) = A^2 - B^2$

 **NOW TRY EXERCISE 81**

EXAMPLE 10 | Rationalizing the Numerator

Rationalize the numerator: $\frac{\sqrt{4 + h} - 2}{h}$

SOLUTION We multiply numerator and denominator by the conjugate radical $\sqrt{4 + h} + 2$.

$$\begin{aligned}\frac{\sqrt{4 + h} - 2}{h} &= \frac{\sqrt{4 + h} - 2}{h} \cdot \frac{\sqrt{4 + h} + 2}{\sqrt{4 + h} + 2} && \text{Multiply numerator and denominator by the conjugate radical} \\ &= \frac{(\sqrt{4 + h})^2 - 2^2}{h(\sqrt{4 + h} + 2)} && \text{Special Product Formula 1} \\ &= \frac{4 + h - 4}{h(\sqrt{4 + h} + 2)} \\ &= \frac{h}{h(\sqrt{4 + h} + 2)} = \frac{1}{\sqrt{4 + h} + 2} && \text{Property 5 of fractions (cancel common factors)}\end{aligned}$$

Special Product Formula 1
 $(A + B)(A - B) = A^2 - B^2$

 **NOW TRY EXERCISE 87**

▼ Avoiding Common Errors



Don't make the mistake of applying properties of multiplication to the operation of addition. Many of the common errors in algebra involve doing just that. The following table states several properties of multiplication and illustrates the error in applying them to addition.

Correct multiplication property	Common error with addition
$(a \cdot b)^2 = a^2 \cdot b^2$	$(a + b)^2 \neq a^2 + b^2$
$\sqrt{a \cdot b} = \sqrt{a} \sqrt{b} \quad (a, b \geq 0)$	$\sqrt{a + b} \neq \sqrt{a} + \sqrt{b}$
$\sqrt{a^2 \cdot b^2} = a \cdot b \quad (a, b \geq 0)$	$\sqrt{a^2 + b^2} \neq a + b$
$\frac{1}{a} \cdot \frac{1}{b} = \frac{1}{a \cdot b}$	$\frac{1}{a} + \frac{1}{b} \neq \frac{1}{a + b}$
$\frac{ab}{a} = b$	$\frac{a + b}{a} \neq b$
$a^{-1} \cdot b^{-1} = (a \cdot b)^{-1}$	$a^{-1} + b^{-1} \neq (a + b)^{-1}$

To verify that the equations in the right-hand column are wrong, simply substitute numbers for a and b and calculate each side. For example, if we take $a = 2$ and $b = 2$ in the fourth error, we find that the left-hand side is

$$\frac{1}{a} + \frac{1}{b} = \frac{1}{2} + \frac{1}{2} = 1$$

whereas the right-hand side is

$$\frac{1}{a + b} = \frac{1}{2 + 2} = \frac{1}{4}$$

Since $1 \neq \frac{1}{4}$, the stated equation is wrong. You should similarly convince yourself of the error in each of the other equations. (See Exercise 105.)

1.4 EXERCISES

CONCEPTS

1. Which of the following are rational expressions?

(a) $\frac{3x}{x^2 - 1}$ (b) $\frac{\sqrt{x+1}}{2x+3}$ (c) $\frac{x(x^2 - 1)}{x+3}$

2. To simplify a rational expression, we cancel *factors* that are common to the _____ and _____. So the expression

$$\frac{(x+1)(x+2)}{(x+3)(x+2)}$$

simplifies to _____.

3. To multiply two rational expressions, we multiply their _____ together and multiply their _____ together.

So $\frac{2}{x+1} \cdot \frac{x}{x+3}$ is the same as _____.

4. Consider the expression $\frac{1}{x} - \frac{2}{x+1} - \frac{x}{(x+1)^2}$.

- (a) How many terms does this expression have?
 (b) Find the least common denominator of all the terms.
 (c) Perform the addition and simplify.

SKILLS

- 5–12 ■ Find the domain of the expression.

5. $4x^2 - 10x + 3$

6. $-x^4 + x^3 + 9x$

7. $\frac{2x+1}{x-4}$

8. $\frac{2t^2-5}{3t+6}$

9. $\sqrt{x+3}$

10. $\frac{1}{\sqrt{x-1}}$

11. $\frac{x^2+1}{x^2-x-2}$

12. $\frac{\sqrt{2x}}{x+1}$

13–22 ■ Simplify the rational expression.

13. $\frac{3(x+2)(x-1)}{6(x-1)^2}$

14. $\frac{4(x^2-1)}{12(x+2)(x-1)}$

15. $\frac{x-2}{x^2-4}$

16. $\frac{x^2-x-2}{x^2-1}$

17. $\frac{x^2+6x+8}{x^2+5x+4}$

18. $\frac{x^2-x-12}{x^2+5x+6}$

19. $\frac{y^2+y}{y^2-1}$

20. $\frac{y^2-3y-18}{2y^2+5y+3}$

21. $\frac{2x^3-x^2-6x}{2x^2-7x+6}$

22. $\frac{1-x^2}{x^3-1}$

23–38 ■ Perform the multiplication or division and simplify.

23. $\frac{4x}{x^2-4} \cdot \frac{x+2}{16x}$

24. $\frac{x^2-25}{x^2-16} \cdot \frac{x+4}{x+5}$

25. $\frac{x^2-2x-15}{x^2-9} \cdot \frac{x+3}{x-5}$

26. $\frac{x^2+2x-3}{x^2-2x-3} \cdot \frac{3-x}{3+x}$

27. $\frac{t-3}{t^2+9} \cdot \frac{t+3}{t^2-9}$

28. $\frac{x^2-x-6}{x^2+2x} \cdot \frac{x^3+x^2}{x^2-2x-3}$

29. $\frac{x^2+7x+12}{x^2+3x+2} \cdot \frac{x^2+5x+6}{x^2+6x+9}$

30. $\frac{x^2+2xy+y^2}{x^2-y^2} \cdot \frac{2x^2-xy-y^2}{x^2-xy-2y^2}$

31. $\frac{x+3}{4x^2-9} \div \frac{x^2+7x+12}{2x^2+7x-15}$

32. $\frac{2x+1}{2x^2+x-15} \div \frac{6x^2-x-2}{x+3}$

33. $\frac{2x^2+3x+1}{x^2+2x-15} \div \frac{x^2+6x+5}{2x^2-7x+3}$

34. $\frac{4y^2-9}{2y^2+9y-18} \div \frac{2y^2+y-3}{y^2+5y-6}$

35. $\frac{\frac{x^3}{x+1}}{\frac{x}{x^2+2x+1}}$

36. $\frac{\frac{2x^2-3x-2}{x^2-1}}{\frac{2x^2+5x+2}{x^2+x-2}}$

37. $\frac{x/y}{z}$

38. $\frac{x}{y/z}$

39–58 ■ Perform the addition or subtraction and simplify.

39. $2 + \frac{x}{x+3}$

40. $\frac{2x-1}{x+4} - 1$

41. $\frac{1}{x+5} + \frac{2}{x-3}$

42. $\frac{1}{x+1} + \frac{1}{x-1}$

43. $\frac{1}{x+1} - \frac{1}{x+2}$

44. $\frac{x}{x-4} - \frac{3}{x+6}$

45. $\frac{x}{(x+1)^2} + \frac{2}{x+1}$

46. $\frac{5}{2x-3} - \frac{3}{(2x-3)^2}$

47. $u+1 + \frac{u}{u+1}$

48. $\frac{2}{a^2} - \frac{3}{ab} + \frac{4}{b^2}$

49. $\frac{1}{x^2} + \frac{1}{x^2+x}$

50. $\frac{1}{x} + \frac{1}{x^2} + \frac{1}{x^3}$

51. $\frac{2}{x+3} - \frac{1}{x^2+7x+12}$

52. $\frac{x}{x^2-4} + \frac{1}{x-2}$

53. $\frac{1}{x+3} + \frac{1}{x^2-9}$

54. $\frac{x}{x^2+x-2} - \frac{2}{x^2-5x+4}$

55. $\frac{2}{x} + \frac{3}{x-1} - \frac{4}{x^2-x}$

56. $\frac{x}{x^2-x-6} - \frac{1}{x+2} - \frac{2}{x-3}$

57. $\frac{1}{x^2+3x+2} - \frac{1}{x^2-2x-3}$

58. $\frac{1}{x+1} - \frac{2}{(x+1)^2} + \frac{3}{x^2-1}$

59–68 ■ Simplify the compound fractional expression.

59. $\frac{x + \frac{1}{x+2}}{x - \frac{1}{x+2}}$

60. $\frac{1 + \frac{1}{c-1}}{1 - \frac{1}{c-1}}$

61. $\frac{\frac{x+2}{x-1} - \frac{x-3}{x-2}}{x+2}$

62. $\frac{\frac{x-3}{x-4} - \frac{x+2}{x+1}}{x+3}$

63. $\frac{\frac{y}{x} - \frac{x}{y}}{\frac{1}{x^2} - \frac{1}{y^2}}$

64. $x - \frac{y}{\frac{x}{y} + \frac{y}{x}}$

65. $\frac{x^{-2} - y^{-2}}{x^{-1} + y^{-1}}$

66. $\frac{x^{-1} + y^{-1}}{(x+y)^{-1}}$

67. $1 - \frac{1}{1 - \frac{1}{x}}$

68. $1 + \frac{1}{1 + \frac{1}{1+x}}$

69–74 ■ Simplify the fractional expression. (Expressions like these arise in calculus.)

69. $\frac{\frac{1}{1+x+h} - \frac{1}{1+x}}{h}$

70. $\frac{\frac{1}{\sqrt{x+h}} - \frac{1}{\sqrt{x}}}{h}$

71. $\frac{\frac{1}{(x+h)^2} - \frac{1}{x^2}}{h}$

72. $\frac{(x+h)^3 - 7(x+h) - (x^3 - 7x)}{h}$

73. $\sqrt{1 + \left(\frac{x}{\sqrt{1-x^2}}\right)^2}$

74. $\sqrt{1 + \left(x^3 - \frac{1}{4x^3}\right)^2}$

75–80 ■ Simplify the expression. (This type of expression arises in calculus when using the “quotient rule.”)

75. $\frac{3(x+2)^2(x-3)^2 - (x+2)^3(2)(x-3)}{(x-3)^4}$

76. $\frac{2x(x+6)^4 - x^2(4)(x+6)^3}{(x+6)^8}$

77. $\frac{2(1+x)^{1/2} - x(1+x)^{-1/2}}{x+1}$

78. $\frac{(1-x^2)^{1/2} + x^2(1-x^2)^{-1/2}}{1-x^2}$

79. $\frac{3(1+x)^{1/3} - x(1+x)^{-2/3}}{(1+x)^{2/3}}$

80. $\frac{(7-3x)^{1/2} + \frac{3}{2}x(7-3x)^{-1/2}}{7-3x}$

81–86 ■ Rationalize the denominator.

81. $\frac{1}{2-\sqrt{3}}$

82. $\frac{2}{3-\sqrt{5}}$

83. $\frac{2}{\sqrt{2}+\sqrt{7}}$

84. $\frac{1}{\sqrt{x}+1}$

85. $\frac{y}{\sqrt{3}+\sqrt{y}}$

86. $\frac{2(x-y)}{\sqrt{x}-\sqrt{y}}$

87–92 ■ Rationalize the numerator.

87. $\frac{1-\sqrt{5}}{3}$

88. $\frac{\sqrt{3}+\sqrt{5}}{2}$

89. $\frac{\sqrt{r}+\sqrt{2}}{5}$

90. $\frac{\sqrt{x}-\sqrt{x+h}}{h\sqrt{x}\sqrt{x+h}}$

91. $\sqrt{x^2+1}-x$

92. $\sqrt{x+1}-\sqrt{x}$

93–100 ■ State whether the given equation is true for all values of the variables. (Disregard any value that makes a denominator zero.)

93. $\frac{16+a}{16} = 1 + \frac{a}{16}$

94. $\frac{b}{b-c} = 1 - \frac{b}{c}$

95. $\frac{2}{4+x} = \frac{1}{2} + \frac{2}{x}$

96. $\frac{x+1}{y+1} = \frac{x}{y}$

97. $\frac{x}{x+y} = \frac{1}{1+y}$

98. $2\left(\frac{a}{b}\right) = \frac{2a}{2b}$

99. $\frac{-a}{b} = -\frac{a}{b}$

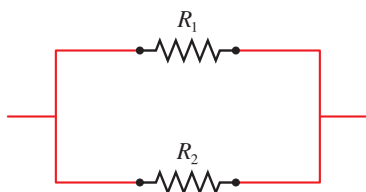
100. $\frac{1+x+x^2}{x} = \frac{1}{x} + 1 + x$

APPLICATIONS

101. Electrical Resistance If two electrical resistors with resistances R_1 and R_2 are connected in parallel (see the figure), then the total resistance R is given by

$$R = \frac{1}{\frac{1}{R_1} + \frac{1}{R_2}}$$

- (a) Simplify the expression for R .
- (b) If $R_1 = 10$ ohms and $R_2 = 20$ ohms, what is the total resistance R ?



- 102. Average Cost** A clothing manufacturer finds that the cost of producing x shirts is $500 + 6x + 0.01x^2$ dollars.
- (a) Explain why the average cost per shirt is given by the rational expression

$$A = \frac{500 + 6x + 0.01x^2}{x}$$

- (b) Complete the table by calculating the average cost per shirt for the given values of x .

x	Average cost
10	
20	
50	
100	
200	
500	
1000	

DISCOVERY ■ DISCUSSION ■ WRITING

- 103. Limiting Behavior of a Rational Expression** The rational expression

$$\frac{x^2 - 9}{x - 3}$$

is not defined for $x = 3$. Complete the tables and determine what value the expression approaches as x gets closer and closer to 3. Why is this reasonable? Factor the numerator of the expression and simplify to see why.

x	$\frac{x^2 - 9}{x - 3}$
2.80	
2.90	
2.95	
2.99	
2.999	

x	$\frac{x^2 - 9}{x - 3}$
3.20	
3.10	
3.05	
3.01	
3.001	

- 104. Is This Rationalization?** In the expression $2/\sqrt{x}$ we would eliminate the radical if we were to square both numerator and denominator. Is this the same thing as rationalizing the denominator?

- 105. Algebraic Errors** The left-hand column in the table on the following page lists some common algebraic errors. In each case, give an example using numbers that show that the formula is not valid. An example of this type, which shows that a statement is false, is called a *counterexample*.

Algebraic error	Counterexample
$\frac{1}{a} + \frac{1}{b} \neq \frac{1}{a+b}$	$\frac{1}{2} + \frac{1}{2} \neq \frac{1}{2+2}$
$(a+b)^2 \neq a^2 + b^2$	
$\sqrt{a^2 + b^2} \neq a + b$	
$\frac{a+b}{a} \neq b$	
$(a^3 + b^3)^{1/3} \neq a + b$	
$a^m / a^n \neq a^{m/n}$	
$a^{-1/n} \neq \frac{1}{a^n}$	

106. The Form of an Algebraic Expression An algebraic expression may look complicated, but its “form” is always simple; it must be a sum, a product, a quotient, or a power. For example, consider the following expressions:

$$(1+x^2)^2 + \left(\frac{x+2}{x+1}\right)^3 \quad (1+x)\left(1 + \frac{x+5}{1+x^4}\right)$$

$$\frac{5-x^3}{1+\sqrt{1+x^2}} \quad \sqrt{\frac{1+x}{1-x}}$$

With appropriate choices for A and B , the first has the form $A + B$, the second AB , the third A/B , and the fourth $A^{1/2}$. Recognizing the form of an expression helps us expand, simplify, or factor it correctly. Find the form of the following algebraic expressions.

(a) $x + \sqrt{1 + \frac{1}{x}}$ (b) $(1+x^2)(1+x)^3$

(c) $\sqrt[3]{x^4(4x^2+1)}$ (d) $\frac{1-2\sqrt{1+x}}{1+\sqrt{1+x^2}}$

1.5 EQUATIONS

Solving Linear Equations ► Solving Quadratic Equations ► Other Types of Equations

An equation is a statement that two mathematical expressions are equal. For example,

$$3 + 5 = 8$$

is an equation. Most equations that we study in algebra contain variables, which are symbols (usually letters) that stand for numbers. In the equation

$$4x + 7 = 19$$

the letter x is the variable. We think of x as the “unknown” in the equation, and our goal is to find the value of x that makes the equation true. The values of the unknown that make the equation true are called the **solutions** or **roots** of the equation, and the process of finding the solutions is called **solving the equation**.

Two equations with exactly the same solutions are called **equivalent equations**. To solve an equation, we try to find a simpler, equivalent equation in which the variable stands alone on one side of the “equal” sign. Here are the properties that we use to solve an equation. (In these properties, A , B , and C stand for any algebraic expressions, and the symbol $\Leftrightarrow$ means “is equivalent to.”)

PROPERTIES OF EQUALITY

Property

1. $A = B \Leftrightarrow A + C = B + C$

2. $A = B \Leftrightarrow CA = CB \quad (C \neq 0)$

Description

Adding the same quantity to both sides of an equation gives an equivalent equation.

Multiplying both sides of an equation by the same nonzero quantity gives an equivalent equation.

$x = 3$ is a solution of the equation $4x + 7 = 19$, because substituting $x = 3$ makes the equation true:

$$x = 3$$

$$4(3) + 7 = 19 \quad \checkmark$$

These properties require that you *perform the same operation on both sides of an equation* when solving it. Thus, if we say “add -7 ” when solving an equation, that is just a short way of saying “add -7 to each side of the equation.”

▼ Solving Linear Equations

The simplest type of equation is a *linear equation*, or first-degree equation, which is an equation in which each term is either a constant or a nonzero multiple of the variable.

LINEAR EQUATIONS

A **linear equation** in one variable is an equation equivalent to one of the form

$$ax + b = 0$$

where a and b are real numbers and x is the variable.

Here are some examples that illustrate the difference between linear and nonlinear equations.

Linear equations

$$4x - 5 = 3$$

$$2x = \frac{1}{2}x - 7$$

$$x - 6 = \frac{x}{3}$$

Nonlinear equations

$$x^2 + 2x = 8$$

$$\sqrt{x} - 6x = 0$$

$$\frac{3}{x} - 2x = 1$$

Not linear; contains the square of the variable

Not linear; contains the square root of the variable

Not linear; contains the reciprocal of the variable

EXAMPLE 1 | Solving a Linear Equation

Solve the equation $7x - 4 = 3x + 8$.

SOLUTION We solve this by changing it to an equivalent equation with all terms that have the variable x on one side and all constant terms on the other.

$$7x - 4 = 3x + 8$$

Given equation

$$(7x - 4) + 4 = (3x + 8) + 4$$

Add 4

$$7x = 3x + 12$$

Simplify

$$7x - 3x = (3x + 12) - 3x$$

Subtract $3x$

$$4x = 12$$

Simplify

$$\frac{1}{4} \cdot 4x = \frac{1}{4} \cdot 12$$

Multiply by $\frac{1}{4}$

$$x = 3$$

Simplify

Because it is important to **CHECK YOUR ANSWER**, we do this in many of our examples. In these checks, LHS stands for “left-hand side” and RHS stands for “right-hand side” of the original equation.

CHECK YOUR ANSWER

$$x = 3:$$

$$\text{LHS} = \text{RHS} \quad \checkmark$$

$$x = 3$$

$$\begin{aligned} \text{LHS} &= 7(3) - 4 \\ &= 17 \end{aligned}$$

$$x = 3$$

$$\begin{aligned} \text{RHS} &= 3(3) + 8 \\ &= 17 \end{aligned}$$

✎ NOW TRY EXERCISE 15

Many formulas in the sciences involve several variables, and it is often necessary to express one of the variables in terms of the others. In the next example we solve for a variable in Newton’s Law of Gravity.

This is Newton's Law of Gravity. It gives the gravitational force F between two masses m and M that are a distance r apart. The constant G is the universal gravitational constant.

EXAMPLE 2 | Solving for One Variable in Terms of Others

Solve for the variable M in the equation

$$F = G \frac{mM}{r^2}$$

SOLUTION Although this equation involves more than one variable, we solve it as usual by isolating M on one side and treating the other variables as we would numbers.

$$F = \left(\frac{Gm}{r^2} \right) M \quad \text{Factor } M \text{ from RHS}$$

$$\left(\frac{r^2}{Gm} \right) F = \left(\frac{r^2}{Gm} \right) \left(\frac{Gm}{r^2} \right) M \quad \text{Multiply by reciprocal of } \frac{Gm}{r^2}$$

$$\frac{r^2 F}{Gm} = M \quad \text{Simplify}$$

The solution is $M = \frac{r^2 F}{Gm}$.

 NOW TRY EXERCISE 29

EXAMPLE 3 | Solving for One Variable in Terms of Others

The surface area A of the closed rectangular box shown in Figure 1 can be calculated from the length l , the width w , and the height h according to the formula

$$A = 2lw + 2wh + 2lh$$

Solve for w in terms of the other variables in this equation.

SOLUTION Although this equation involves more than one variable, we solve it as usual by isolating w on one side, treating the other variables as we would numbers.

$$A = (2lw + 2wh) + 2lh \quad \text{Collect terms involving } w$$

$$A - 2lh = 2lw + 2wh \quad \text{Subtract } 2lh$$

$$A - 2lh = (2l + 2h)w \quad \text{Factor } w \text{ from RHS}$$

$$\frac{A - 2lh}{2l + 2h} = w \quad \text{Divide by } 2l + 2h$$

The solution is $w = \frac{A - 2lh}{2l + 2h}$.

 NOW TRY EXERCISE 31

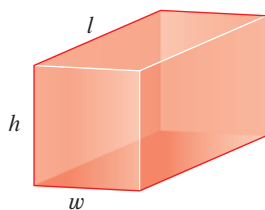


FIGURE 1 A closed rectangular box

▼ Solving Quadratic Equations

Linear equations are first-degree equations like $2x + 1 = 5$ or $4 - 3x = 2$. Quadratic equations are second-degree equations like $x^2 + 2x - 3 = 0$ or $2x^2 + 3 = 5x$.

Quadratic Equations

$$x^2 - 2x - 8 = 0$$

$$3x + 10 = 4x^2$$

$$\frac{1}{2}x^2 + \frac{1}{3}x - \frac{1}{6} = 0$$

QUADRATIC EQUATIONS

A **quadratic equation** is an equation of the form

$$ax^2 + bx + c = 0$$

where a , b , and c are real numbers with $a \neq 0$.

Some quadratic equations can be solved by factoring and using the following basic property of real numbers.

ZERO-PRODUCT PROPERTY

$$AB = 0 \quad \text{if and only if} \quad A = 0 \quad \text{or} \quad B = 0$$



This means that if we can factor the left-hand side of a quadratic (or other) equation, then we can solve it by setting each factor equal to 0 in turn. **This method works only when the right-hand side of the equation is 0.**

EXAMPLE 4 | Solving a Quadratic Equation by Factoring

Solve the equation $x^2 + 5x = 24$.

SOLUTION We must first rewrite the equation so that the right-hand side is 0.

$$x^2 + 5x = 24$$

$$x^2 + 5x - 24 = 0 \quad \text{Subtract 24}$$

$$(x - 3)(x + 8) = 0 \quad \text{Factor}$$

$$x - 3 = 0 \quad \text{or} \quad x + 8 = 0 \quad \text{Zero-Product Property}$$

$$x = 3 \quad \quad \quad x = -8 \quad \text{Solve}$$

CHECK YOUR ANSWERS

$$x = 3:$$

$$(3)^2 + 5(3) = 9 + 15 = 24 \quad \checkmark$$

$$x = -8:$$

$$(-8)^2 + 5(-8) = 64 - 40 = 24 \quad \checkmark$$

The solutions are $x = 3$ and $x = -8$.

NOW TRY EXERCISE 43

Do you see why one side of the equation must be 0 in Example 4? Factoring the equation as $x(x + 5) = 24$ does not help us find the solutions, since 24 can be factored in infinitely many ways, such as $6 \cdot 4$, $\frac{1}{2} \cdot 48$, $(-\frac{2}{5}) \cdot (-60)$, and so on.

A quadratic equation of the form $x^2 - c = 0$, where c is a positive constant, factors as $(x - \sqrt{c})(x + \sqrt{c}) = 0$, so the solutions are $x = \sqrt{c}$ and $x = -\sqrt{c}$. We often abbreviate this as $x = \pm\sqrt{c}$.

SOLVING A SIMPLE QUADRATIC EQUATION

The solutions of the equation $x^2 = c$ are $x = \sqrt{c}$ and $x = -\sqrt{c}$.

EXAMPLE 5 | Solving Simple Quadratics

Solve each equation.

$$(a) \ x^2 = 5 \quad (b) \ (x - 4)^2 = 5$$

SOLUTION

(a) From the principle in the preceding box, we get $x = \pm\sqrt{5}$.

(b) We can take the square root of each side of this equation as well.

$$(x - 4)^2 = 5$$

$$x - 4 = \pm\sqrt{5} \quad \text{Take the square root}$$

$$x = 4 \pm \sqrt{5} \quad \text{Add 4}$$

The solutions are $x = 4 + \sqrt{5}$ and $x = 4 - \sqrt{5}$.

NOW TRY EXERCISES 51 AND 53

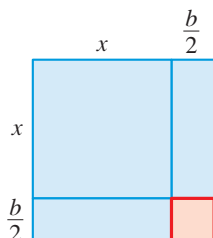
See page 30 for how to recognize when a quadratic expression is a perfect square.

Completing the Square

The area of the blue region is

$$x^2 + 2\left(\frac{b}{2}\right)x = x^2 + bx$$

Add a small square of area $(b/2)^2$ to “complete” the square.



 When completing the square, make sure the coefficient of x^2 is 1. If it isn't, you must factor this coefficient from both terms that contain x :

$$ax^2 + bx = a\left(x^2 + \frac{b}{a}x\right)$$

Then complete the square inside the parentheses. Remember that the term added inside the parentheses is multiplied by a .

As we saw in Example 5, if a quadratic equation is of the form $(x \pm a)^2 = c$, then we can solve it by taking the square root of each side. In an equation of this form the left-hand side is a *perfect square*: the square of a linear expression in x . So if a quadratic equation does not factor readily, then we can solve it using the technique of **completing the square**. This means that we add a constant to an expression to make it a perfect square. For example, to make $x^2 - 6x$ a perfect square, we must add 9, since $x^2 - 6x + 9 = (x - 3)^2$.

COMPLETING THE SQUARE

To make $x^2 + bx$ a perfect square, add $\left(\frac{b}{2}\right)^2$, the square of half the coefficient of x . This gives the perfect square

$$x^2 + bx + \left(\frac{b}{2}\right)^2 = \left(x + \frac{b}{2}\right)^2$$

EXAMPLE 6 | Solving Quadratic Equations by Completing the Square

Solve each equation.

(a) $x^2 - 8x + 13 = 0$

(b) $3x^2 - 12x + 6 = 0$

SOLUTION

(a) $x^2 - 8x + 13 = 0$

Given equation

$$x^2 - 8x = -13$$

Subtract 13

$$x^2 - 8x + 16 = -13 + 16$$

Complete the square: add $\left(\frac{-8}{2}\right)^2 = 16$

$$(x - 4)^2 = 3$$

Perfect square

$$x - 4 = \pm \sqrt{3}$$

Take square root

$$x = 4 \pm \sqrt{3}$$

Add 4

(b) After subtracting 6 from each side of the equation, we must factor the coefficient of x^2 (the 3) from the left side to put the equation in the correct form for completing the square.

$$3x^2 - 12x + 6 = 0$$

Given equation

$$3x^2 - 12x = -6$$

Subtract 6

$$3(x^2 - 4x) = -6$$

Factor 3 from LHS

Now we complete the square by adding $(-2)^2 = 4$ inside the parentheses. Since everything inside the parentheses is multiplied by 3, this means that we are actually adding $3 \cdot 4 = 12$ to the left side of the equation. Thus, we must add 12 to the right side as well.

$$3(x^2 - 4x + 4) = -6 + 3 \cdot 4$$

Complete the square: add 4

$$3(x - 2)^2 = 6$$

Perfect square

$$(x - 2)^2 = 2$$

Divide by 3

$$x - 2 = \pm \sqrt{2}$$

Take square root

$$x = 2 \pm \sqrt{2}$$

Add 2

 NOW TRY EXERCISES 55 AND 59



Library of Congress

FRANÇOIS VIÈTE (1540–1603) had a successful political career before taking up mathematics late in life. He became one of the most famous French mathematicians of the 16th century. Viète introduced a new level of abstraction in algebra by using letters to stand for *known* quantities in an equation. Before Viète's time, each equation had to be solved on its own. For instance, the quadratic equations

$$3x^2 + 2x + 8 = 0$$

$$5x^2 - 6x + 4 = 0$$

had to be solved separately by completing the square. Viète's idea was to consider all quadratic equations at once by writing

$$ax^2 + bx + c = 0$$

where a , b , and c are known quantities. Thus, he made it possible to write a *formula* (in this case, the quadratic formula) involving a , b , and c that can be used to solve all such equations in one fell swoop.

Viète's mathematical genius proved quite valuable during a war between France and Spain. To communicate with their troops, the Spaniards used a complicated code that Viète managed to decipher. Unaware of Viète's accomplishment, the Spanish king, Philip II, protested to the Pope, claiming that the French were using witchcraft to read his messages.

Another Method

$$\begin{aligned} 4x^2 + 12x + 9 &= 0 \\ (2x + 3)^2 &= 0 \\ 2x + 3 &= 0 \\ x &= -\frac{3}{2} \end{aligned}$$

We can use the technique of completing the square to derive a formula for the roots of the general quadratic equation $ax^2 + bx + c = 0$.

THE QUADRATIC FORMULA

The roots of the quadratic equation $ax^2 + bx + c = 0$, where $a \neq 0$, are

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

PROOF First, we divide each side of the equation by a and move the constant to the right side, giving

$$x^2 + \frac{b}{a}x = -\frac{c}{a} \quad \text{Divide by } a$$

We now complete the square by adding $(b/2a)^2$ to each side of the equation:

$$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = -\frac{c}{a} + \left(\frac{b}{2a}\right)^2 \quad \text{Complete the square: Add } \left(\frac{b}{2a}\right)^2$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{-4ac + b^2}{4a^2} \quad \text{Perfect square}$$

$$x + \frac{b}{2a} = \pm \frac{\sqrt{b^2 - 4ac}}{2a} \quad \text{Take square root}$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad \text{Subtract } \frac{b}{2a}$$

The Quadratic Formula could be used to solve the equations in Examples 4 and 6. You should carry out the details of these calculations.

EXAMPLE 7 | Using the Quadratic Formula

Find all solutions of each equation.

(a) $3x^2 - 5x - 1 = 0$ (b) $4x^2 + 12x + 9 = 0$ (c) $x^2 + 2x + 2 = 0$

SOLUTION

(a) In this quadratic equation $a = 3$, $b = -5$, and $c = -1$.

$$\begin{array}{c} b = -5 \\ 3x^2 - 5x - 1 = 0 \\ \begin{array}{cc} \nearrow & \nwarrow \\ a = 3 & c = -1 \end{array} \end{array}$$

By the Quadratic Formula,

$$x = \frac{-(-5) \pm \sqrt{(-5)^2 - 4(3)(-1)}}{2(3)} = \frac{5 \pm \sqrt{37}}{6}$$

If approximations are desired, we can use a calculator to obtain

$$x = \frac{5 + \sqrt{37}}{6} \approx 1.8471 \quad \text{and} \quad x = \frac{5 - \sqrt{37}}{6} \approx -0.1805$$

(b) Using the Quadratic Formula with $a = 4$, $b = 12$, and $c = 9$ gives

$$x = \frac{-12 \pm \sqrt{(12)^2 - 4 \cdot 4 \cdot 9}}{2 \cdot 4} = \frac{-12 \pm 0}{8} = -\frac{3}{2}$$

This equation has only one solution, $x = -\frac{3}{2}$.

- (c) Using the Quadratic Formula with $a = 1$, $b = 2$, and $c = 2$ gives

$$x = \frac{-2 \pm \sqrt{2^2 - 4 \cdot 2}}{2} = \frac{-2 \pm \sqrt{-4}}{2} = \frac{-2 \pm 2\sqrt{-1}}{2} = -1 \pm \sqrt{-1}$$

Since the square of any real number is nonnegative, $\sqrt{-1}$ is undefined in the real number system. The equation has no real solution.

 NOW TRY EXERCISES 65, 69, AND 75

In Section 3.5 we study the complex number system, in which the square roots of negative numbers do exist. The equation in Example 7(c) does have solutions in the complex number system.

The quantity $b^2 - 4ac$ that appears under the square root sign in the quadratic formula is called the *discriminant* of the equation $ax^2 + bx + c = 0$ and is given the symbol D . If $D < 0$, then $\sqrt{b^2 - 4ac}$ is undefined, and the quadratic equation has no real solution, as in Example 7(c). If $D = 0$, then the equation has only one real solution, as in Example 7(b). Finally, if $D > 0$, then the equation has two distinct real solutions, as in Example 7(a). The following box summarizes these observations.

THE DISCRIMINANT

The **discriminant** of the general quadratic $ax^2 + bx + c = 0$ ($a \neq 0$) is $D = b^2 - 4ac$.

1. If $D > 0$, then the equation has two distinct real solutions.
2. If $D = 0$, then the equation has exactly one real solution.
3. If $D < 0$, then the equation has no real solution.

EXAMPLE 8 | Using the Discriminant

Use the discriminant to determine how many real solutions each equation has.

- (a) $x^2 + 4x - 1 = 0$ (b) $4x^2 - 12x + 9 = 0$ (c) $\frac{1}{3}x^2 - 2x + 4 = 0$

SOLUTION

- (a) The discriminant is $D = 4^2 - 4(1)(-1) = 20 > 0$, so the equation has two distinct real solutions.
- (b) The discriminant is $D = (-12)^2 - 4 \cdot 4 \cdot 9 = 0$, so the equation has exactly one real solution.
- (c) The discriminant is $D = (-2)^2 - 4(\frac{1}{3})4 = -\frac{4}{3} < 0$, so the equation has no real solution.

 NOW TRY EXERCISES 79, 81, AND 83

Now let's consider a real-life situation that can be modeled by a quadratic equation.

EXAMPLE 9 | The Path of a Projectile

An object thrown or fired straight upward at an initial speed of v_0 ft/s will reach a height of h feet after t seconds, where h and t are related by the formula

$$h = -16t^2 + v_0t$$

Suppose that a bullet is shot straight upward with an initial speed of 800 ft/s. Its path is shown in Figure 2.

- (a) When does the bullet fall back to ground level?
- (b) When does it reach a height of 6400 ft?

This formula depends on the fact that acceleration due to gravity is constant near the earth's surface. Here we neglect the effect of air resistance.

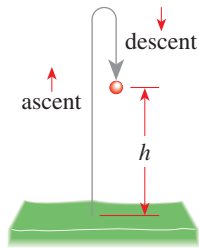
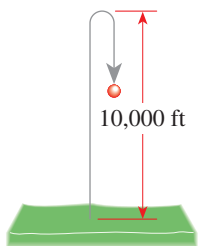
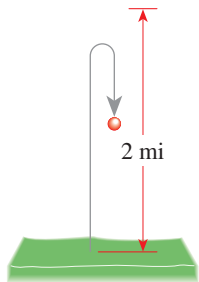
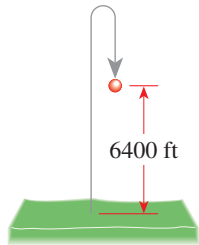
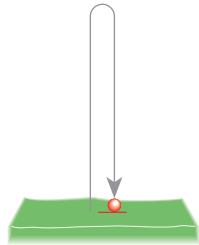


FIGURE 2



- (c) When does it reach a height of 2 mi?
 (d) How high is the highest point the bullet reaches?

SOLUTION Since the initial speed in this case is $v_0 = 800$ ft/s, the formula is

$$h = -16t^2 + 800t$$

- (a) Ground level corresponds to $h = 0$, so we must solve the equation

$$0 = -16t^2 + 800t \quad \text{Set } h = 0$$

$$0 = -16t(t - 50) \quad \text{Factor}$$

Thus, $t = 0$ or $t = 50$. This means the bullet starts ($t = 0$) at ground level and returns to ground level after 50 s.

- (b) Setting $h = 6400$ gives the equation

$$6400 = -16t^2 + 800t \quad \text{Set } h = 6400$$

$$16t^2 - 800t + 6400 = 0 \quad \text{All terms to LHS}$$

$$t^2 - 50t + 400 = 0 \quad \text{Divide by 16}$$

$$(t - 10)(t - 40) = 0 \quad \text{Factor}$$

$$t = 10 \quad \text{or} \quad t = 40 \quad \text{Solve}$$

The bullet reaches 6400 ft after 10 s (on its ascent) and again after 40 s (on its descent to earth).

- (c) Two miles is $2 \times 5280 = 10,560$ ft.

$$10,560 = -16t^2 + 800t \quad \text{Set } h = 10,560$$

$$16t^2 - 800t + 10,560 = 0 \quad \text{All terms to LHS}$$

$$t^2 - 50t + 660 = 0 \quad \text{Divide by 16}$$

The discriminant of this equation is $D = (-50)^2 - 4(660) = -140$, which is negative. Thus, the equation has no real solution. The bullet never reaches a height of 2 mi.

- (d) Each height the bullet reaches is attained twice, once on its ascent and once on its descent. The only exception is the highest point of its path, which is reached only once. This means that for the highest value of h , the following equation has only one solution for t :

$$h = -16t^2 + 800t$$

$$16t^2 - 800t + h = 0 \quad \text{All terms to LHS}$$

This in turn means that the discriminant D of the equation is 0, so

$$D = (-800)^2 - 4(16)h = 0$$

$$640,000 - 64h = 0$$

$$h = 10,000$$

The maximum height reached is 10,000 ft.

 **NOW TRY EXERCISE 111**

Other Types of Equations

So far we have learned how to solve linear and quadratic equations. Now we study other types of equations, including those that involve higher powers, fractional expressions, and radicals.

EXAMPLE 10 | An Equation Involving Fractional ExpressionsSolve the equation $\frac{3}{x} + \frac{5}{x+2} = 2$.**SOLUTION** We eliminate the denominators by multiplying each side by the lowest common denominator.

$$\left(\frac{3}{x} + \frac{5}{x+2}\right)x(x+2) = 2x(x+2) \quad \text{Multiply by LCD } x(x+2)$$

$$3(x+2) + 5x = 2x^2 + 4x \quad \text{Expand}$$

$$8x + 6 = 2x^2 + 4x \quad \text{Expand LHS}$$

$$0 = 2x^2 - 4x - 6 \quad \text{Subtract } 8x + 6$$

$$0 = x^2 - 2x - 3 \quad \text{Divide both sides by 2}$$

$$0 = (x-3)(x+1) \quad \text{Factor}$$

$$x-3=0 \quad \text{or} \quad x+1=0 \quad \text{Zero-Product Property}$$

$$x=3 \quad \quad \quad x=-1 \quad \text{Solve}$$

We must check our answers because multiplying by an expression that contains the variable can introduce extraneous solutions. From *Check Your Answers* we see that the solutions are $x = 3$ and -1 .

 NOW TRY EXERCISE 85

When you solve an equation that involves radicals, you must be especially careful to check your final answers. The next example demonstrates why.

EXAMPLE 11 | An Equation Involving a RadicalSolve the equation $2x = 1 - \sqrt{2-x}$.**SOLUTION** To eliminate the square root, we first isolate it on one side of the equation, then square:

$$2x - 1 = -\sqrt{2-x} \quad \text{Subtract 1}$$

$$(2x-1)^2 = 2-x \quad \text{Square each side}$$

$$4x^2 - 4x + 1 = 2-x \quad \text{Expand LHS}$$

$$4x^2 - 3x - 1 = 0 \quad \text{Add } -2 + x$$

$$(4x+1)(x-1) = 0 \quad \text{Factor}$$

$$4x+1=0 \quad \text{or} \quad x-1=0 \quad \text{Zero-Product Property}$$

$$x = -\frac{1}{4} \quad \quad \quad x = 1 \quad \text{Solve}$$

The values $x = -\frac{1}{4}$ and $x = 1$ are only potential solutions. We must check them to see if they satisfy the original equation. From *Check Your Answers* we see that $x = -\frac{1}{4}$ is a solution but $x = 1$ is not. The only solution is $x = -\frac{1}{4}$.

 NOW TRY EXERCISE 91

When we solve an equation, we may end up with one or more **extraneous solutions**, that is, potential solutions that do not satisfy the original equation. In Example 11 the value $x = 1$ is an extraneous solution. Extraneous solutions may be introduced when we square each side of an equation because the operation of squaring can turn a false equation into a true one. For example, $-1 \neq 1$, but $(-1)^2 = 1^2$. Thus, the squared equation

CHECK YOUR ANSWERS $x = 3$:

$$\text{LHS} = \frac{3}{3} + \frac{5}{3+2}$$

$$= 1 + 1 = 2$$

$$\text{RHS} = 2$$

$$\text{LHS} = \text{RHS} \quad \checkmark$$

 $x = -1$:

$$\text{LHS} = \frac{3}{-1} + \frac{5}{-1+2}$$

$$= -3 + 5 = 2$$

$$\text{RHS} = 2$$

$$\text{LHS} = \text{RHS} \quad \checkmark$$

CHECK YOUR ANSWERS $x = -\frac{1}{4}$:

$$\text{LHS} = 2\left(-\frac{1}{4}\right) = -\frac{1}{2}$$

$$\text{RHS} = 1 - \sqrt{2 - \left(-\frac{1}{4}\right)}$$

$$= 1 - \sqrt{\frac{9}{4}}$$

$$= 1 - \frac{3}{2} = -\frac{1}{2}$$

$$\text{LHS} = \text{RHS} \quad \checkmark$$

 $x = 1$:

$$\text{LHS} = 2(1) = 2$$

$$\text{RHS} = 1 - \sqrt{2-1}$$

$$= 1 - 1 = 0$$

$$\text{LHS} \neq \text{RHS} \quad \times$$



may be true for more values of the variable than the original equation. **That is why you must always check your answers to make sure that each satisfies the original equation.**

An equation of the form $aW^2 + bW + c = 0$, where W is an algebraic expression, is an equation of **quadratic type**. We solve equations of quadratic type by substituting for the algebraic expression, as we see in the next two examples.

EXAMPLE 12 | A Fourth-Degree Equation of Quadratic Type

Find all solutions of the equation $x^4 - 8x^2 + 8 = 0$.

SOLUTION If we set $W = x^2$, then we get a quadratic equation in the new variable W :

$$(x^2)^2 - 8x^2 + 8 = 0 \quad \text{Write } x^4 \text{ as } (x^2)^2$$

$$W^2 - 8W + 8 = 0 \quad \text{Let } W = x^2$$

$$W = \frac{-(-8) \pm \sqrt{(-8)^2 - 4 \cdot 8}}{2} = 4 \pm 2\sqrt{2} \quad \text{Quadratic Formula}$$

$$x^2 = 4 \pm 2\sqrt{2} \quad \text{W} = x^2$$

$$x = \pm \sqrt{4 \pm 2\sqrt{2}} \quad \text{Take square roots}$$

So, there are four solutions:

$$\sqrt{4 + 2\sqrt{2}}, \quad \sqrt{4 - 2\sqrt{2}}, \quad -\sqrt{4 + 2\sqrt{2}}, \quad -\sqrt{4 - 2\sqrt{2}}$$

Using a calculator, we obtain the approximations $x \approx 2.61, 1.08, -2.61, -1.08$.

NOW TRY EXERCISE 95

EXAMPLE 13 | An Equation Involving Fractional Powers

Find all solutions of the equation $x^{1/3} + x^{1/6} - 2 = 0$.

SOLUTION This equation is of quadratic type because if we let $W = x^{1/6}$, then $W^2 = (x^{1/6})^2 = x^{1/3}$.

$$x^{1/3} + x^{1/6} - 2 = 0$$

$$W^2 + W - 2 = 0 \quad \text{Let } W = x^{1/6}$$

$$(W - 1)(W + 2) = 0 \quad \text{Factor}$$

$$W - 1 = 0 \quad \text{or} \quad W + 2 = 0 \quad \text{Zero-Product Property}$$

$$W = 1 \quad \text{or} \quad W = -2 \quad \text{Solve}$$

$$x^{1/6} = 1 \quad \text{or} \quad x^{1/6} = -2 \quad \text{W} = x^{1/6}$$

$$x = 1^6 = 1 \quad \text{or} \quad x = (-2)^6 = 64 \quad \text{Take the 6th power}$$

From *Check Your Answers* we see that $x = 1$ is a solution but $x = 64$ is not. The only solution is $x = 1$.

CHECK YOUR ANSWERS

$x = 1$:

$$\text{LHS} = 1^{1/3} + 1^{1/6} - 2 = 0$$

$$\text{RHS} = 0$$

$$\text{LHS} = \text{RHS} \quad \checkmark$$

$x = 64$:

$$\text{LHS} = 64^{1/3} + 64^{1/6} - 2$$

$$= 4 + 2 - 2 = 4$$

$$\text{RHS} = 0$$

$$\text{LHS} \neq \text{RHS} \quad \times$$

NOW TRY EXERCISE 99

When solving equations that involve absolute values, we usually take cases.

EXAMPLE 14 | An Absolute Value Equation

Solve the equation $|2x - 5| = 3$.

SOLUTION By the definition of absolute value, $|2x - 5| = 3$ is equivalent to

$$2x - 5 = 3 \quad \text{or} \quad 2x - 5 = -3$$

$$2x = 8 \qquad \qquad 2x = 2$$

$$x = 4 \qquad \qquad x = 1$$

The solutions are $x = 1, x = 4$.

 **NOW TRY EXERCISE 105**

1.5 EXERCISES

CONCEPTS

- True or false?
 - Adding the same number to each side of an equation always gives an equivalent equation.
 - Multiplying each side of an equation by the same number always gives an equivalent equation.
 - Squaring each side of an equation always gives an equivalent equation.
- Explain how you would use each method to solve the equation $x^2 - 4x - 5 = 0$.
 - By factoring: _____
 - By completing the square: _____
 - By using the Quadratic Formula: _____
- The solutions of the equation $x^2(x - 4) = 0$ are _____.
 - To solve the equation $x^3 - 4x^2 = 0$, we _____ the left-hand side.
- Solve the equation $\sqrt{2x} + x = 0$ by doing the following steps.
 - Isolate the radical: _____.
 - Square both sides: _____.
 - The solutions of the resulting quadratic equation are _____.
 - The solution(s) that satisfy the original equation are _____.
- The equation $(x + 1)^2 - 5(x + 1) + 6 = 0$ is of _____ type. To solve the equation, we set $W =$ _____. The resulting quadratic equation is _____.
- The equation $x^6 + 7x^3 - 8 = 0$ is of _____ type. To solve the equation, we set $W =$ _____. The resulting quadratic equation is _____.

SKILLS

7–10 ■ Determine whether the given value is a solution of the equation.

7. $4x + 7 = 9x - 3$

(a) $x = -2$ (b) $x = 2$

8. $1 - [2 - (3 - x)] = 4x - (6 + x)$

(a) $x = 2$ (b) $x = 4$

9. $\frac{1}{x} - \frac{1}{x-4} = 1$

(a) $x = 2$ (b) $x = 4$

10. $\frac{x^{3/2}}{x-6} = x-8$

(a) $x = 4$ (b) $x = 8$

11–28 ■ The given equation is either linear or equivalent to a linear equation. Solve the equation.

11. $2x + 7 = 31$

12. $5x - 3 = 4$

13. $\frac{1}{2}x - 8 = 1$

14. $3 + \frac{1}{3}x = 5$

 15. $-7w = 15 - 2w$

16. $5t - 13 = 12 - 5t$

17. $\frac{1}{2}y - 2 = \frac{1}{3}y$

18. $\frac{z}{5} = \frac{3}{10}z + 7$

19. $2(1 - x) = 3(1 + 2x) + 5$

20. $\frac{2}{3}y + \frac{1}{2}(y - 3) = \frac{y + 1}{4}$

21. $x - \frac{1}{3}x - \frac{1}{2}x - 5 = 0$

22. $2x - \frac{x}{2} + \frac{x+1}{4} = 6x$

23. $\frac{1}{x} = \frac{4}{3x} + 1$

24. $\frac{2x-1}{x+2} = \frac{4}{5}$

25. $\frac{3}{x+1} - \frac{1}{2} = \frac{1}{3x+3}$

26. $\frac{4}{x-1} + \frac{2}{x+1} = \frac{35}{x^2-1}$

27. $(t-4)^2 = (t+4)^2 + 32$

28. $\sqrt{3}x + \sqrt{12} = \frac{x+5}{\sqrt{3}}$

29–42 ■ Solve the equation for the indicated variable.

 29. $PV = nRT$; for R

30. $F = G \frac{mM}{r^2}$; for m

31. $P = 2l + 2w$; for w 32. $\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$; for R_1

33. $\frac{ax + b}{cx + d} = 2$; for x

34. $a - 2[b - 3(c - x)] = 6$; for x

35. $a^2x + (a - 1) = (a + 1)x$; for x

36. $\frac{a + 1}{b} = \frac{a - 1}{b} + \frac{b + 1}{a}$; for a

37. $V = \frac{1}{3}\pi r^2 h$; for r 38. $F = G \frac{mM}{r^2}$; for r

39. $a^2 + b^2 = c^2$; for b

40. $A = P \left(1 + \frac{i}{100} \right)^2$; for i

41. $h = \frac{1}{2}gt^2 + v_0t$; for t 42. $S = \frac{n(n + 1)}{2}$; for n

43–54 ■ Solve the equation by factoring.

43. $x^2 + x - 12 = 0$

44. $x^2 + 3x - 4 = 0$

45. $x^2 - 7x + 12 = 0$

46. $x^2 + 8x + 12 = 0$

47. $4x^2 - 4x - 15 = 0$

48. $2y^2 + 7y + 3 = 0$

49. $3x^2 + 5x = 2$

50. $6x(x - 1) = 21 - x$

51. $2x^2 = 8$

52. $3x^2 - 27 = 0$

53. $(3x + 2)^2 = 10$

54. $(2x - 1)^2 = 8$

55–62 ■ Solve the equation by completing the square.

55. $x^2 + 2x - 5 = 0$

56. $x^2 - 4x + 2 = 0$

57. $x^2 - 6x - 11 = 0$

58. $x^2 + 3x - \frac{7}{4} = 0$

59. $2x^2 + 8x + 1 = 0$

60. $3x^2 - 6x - 1 = 0$

61. $4x^2 - x = 0$

62. $x^2 = \frac{3}{4}x - \frac{1}{8}$

63–78 ■ Find all real solutions of the quadratic equation.

63. $x^2 - 2x - 15 = 0$

64. $x^2 + 5x - 6 = 0$

65. $x^2 - 7x + 10 = 0$

66. $x^2 + 30x + 200 = 0$

67. $2x^2 + x - 3 = 0$

68. $3x^2 + 7x + 4 = 0$

69. $3x^2 + 6x - 5 = 0$

70. $x^2 - 6x + 1 = 0$

71. $z^2 - \frac{3}{2}z + \frac{9}{16} = 0$

72. $2y^2 - y - \frac{1}{2} = 0$

73. $4x^2 + 16x - 9 = 0$

74. $0 = x^2 - 4x + 1$

75. $w^2 = 3(w - 1)$

76. $3 + 5z + z^2 = 0$

77. $10y^2 - 16y + 5 = 0$

78. $25x^2 + 70x + 49 = 0$

79–84 ■ Use the discriminant to determine the number of real solutions of the equation. Do not solve the equation.

79. $x^2 - 6x + 1 = 0$

80. $3x^2 = 6x - 9$

81. $x^2 + 2.20x + 1.21 = 0$

82. $x^2 + 2.21x + 1.21 = 0$

83. $4x^2 + 5x + \frac{13}{8} = 0$

84. $x^2 + rx - s = 0 \quad (s > 0)$

85–108 ■ Find all real solutions of the equation.

85. $\frac{1}{x-1} + \frac{1}{x+2} = \frac{5}{4}$

86. $\frac{10}{x} - \frac{12}{x-3} + 4 = 0$

87. $\frac{x^2}{x+100} = 50$

88. $\frac{1}{x-1} - \frac{2}{x^2} = 0$

89. $\frac{x+5}{x-2} = \frac{5}{x+2} + \frac{28}{x^2-4}$

90. $\frac{x}{2x+7} - \frac{x+1}{x+3} = 1$

91. $\sqrt{2x+1} + 1 = x$

92. $\sqrt{5-x} + 1 = x - 2$

93. $2x + \sqrt{x+1} = 8$

94. $\sqrt{\sqrt{x-5} + x} = 5$

95. $x^4 - 13x^2 + 40 = 0$

96. $x^4 - 5x^2 + 4 = 0$

97. $2x^4 + 4x^2 + 1 = 0$

98. $x^6 - 2x^3 - 3 = 0$

99. $x^{4/3} - 5x^{2/3} + 6 = 0$

100. $\sqrt{x} - 3\sqrt[4]{x} - 4 = 0$

101. $4(x+1)^{1/2} - 5(x+1)^{3/2} + (x+1)^{5/2} = 0$

102. $x^{1/2} + 3x^{-1/2} = 10x^{-3/2}$

103. $x^{1/2} - 3x^{1/3} = 3x^{1/6} - 9$

104. $x - 5\sqrt{x} + 6 = 0$

105. $|3x + 5| = 1$

106. $|2x| = 3$

107. $|x - 4| = 0.01$

108. $|x - 6| = -1$

APPLICATIONS

109–110 ■ Falling-Body Problems Suppose an object is dropped from a height h_0 above the ground. Then its height after t seconds is given by $h = -16t^2 + h_0$, where h is measured in feet. Use this information to solve the problem.

109. If a ball is dropped from 288 ft above the ground, how long does it take to reach ground level?

110. A ball is dropped from the top of a building 96 ft tall.

(a) How long will it take to fall half the distance to ground level?

(b) How long will it take to fall to ground level?

111–112 ■ Falling-Body Problems Use the formula $h = -16t^2 + v_0t$ discussed in Example 9.

111. A ball is thrown straight upward at an initial speed of $v_0 = 40$ ft/s.

(a) When does the ball reach a height of 24 ft?

(b) When does it reach a height of 48 ft?

(c) What is the greatest height reached by the ball?

(d) When does the ball reach the highest point of its path?

(e) When does the ball hit the ground?

112. How fast would a ball have to be thrown upward to reach a maximum height of 100 ft? [Hint: Use the discriminant of the equation $16t^2 - v_0t + h = 0$.]

113. Shrinkage in Concrete Beams As concrete dries, it shrinks—the higher the water content, the greater the shrinkage. If a concrete beam has a water content of w kg/m³, then it will shrink by a factor

$$S = \frac{0.032w - 2.5}{10,000}$$

where S is the fraction of the original beam length that disappears due to shrinkage.

(a) A beam 12.025 m long is cast in concrete that contains 250 kg/m³ water. What is the shrinkage factor S ? How long will the beam be when it has dried?

- (b) A beam is 10.014 m long when wet. We want it to shrink to 10.009 m, so the shrinkage factor should be $S = 0.00050$. What water content will provide this amount of shrinkage?



- 114. The Lens Equation** If F is the focal length of a convex lens and an object is placed at a distance x from the lens, then its image will be at a distance y from the lens, where F , x , and y are related by the *lens equation*

$$\frac{1}{F} = \frac{1}{x} + \frac{1}{y}$$

Suppose that a lens has a focal length of 4.8 cm and that the image of an object is 4 cm closer to the lens than the object itself. How far from the lens is the object?

- 115. Fish Population** The fish population in a certain lake rises and falls according to the formula

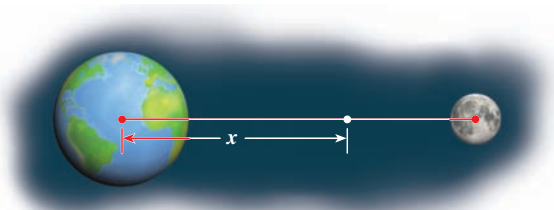
$$F = 1000(30 + 17t - t^2)$$

Here F is the number of fish at time t , where t is measured in years since January 1, 2002, when the fish population was first estimated.

- (a) On what date will the fish population again be the same as it was on January 1, 2002?
- (b) By what date will all the fish in the lake have died?
- 116. Fish Population** A large pond is stocked with fish. The fish population P is modeled by the formula $P = 3t + 10\sqrt{t} + 140$, where t is the number of days since the fish were first introduced into the pond. How many days will it take for the fish population to reach 500?
- 117. Profit** A small-appliance manufacturer finds that the profit P (in dollars) generated by producing x microwave ovens per week is given by the formula $P = \frac{1}{10}x(300 - x)$ provided that $0 \leq x \leq 200$. How many ovens must be manufactured in a given week to generate a profit of \$1250?
- 118. Gravity** If an imaginary line segment is drawn between the centers of the earth and the moon, then the net gravitational force F acting on an object situated on this line segment is

$$F = \frac{-K}{x^2} + \frac{0.012K}{(239 - x)^2}$$

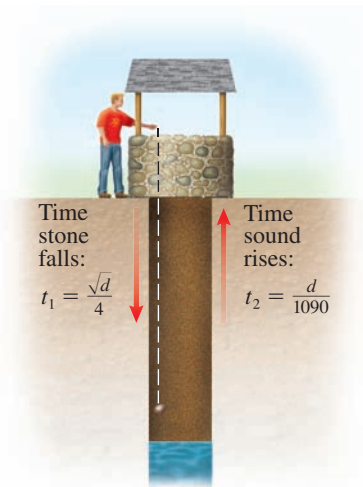
where $K > 0$ is a constant and x is the distance of the object from the center of the earth, measured in thousands of miles. How far from the center of the earth is the “dead spot” where no net gravitational force acts upon the object? (Express your answer to the nearest thousand miles.)



- 119. Depth of a Well** One method for determining the depth of a well is to drop a stone into it and then measure the time it takes until the splash is heard. If d is the depth of the well (in feet) and t_1 the time (in seconds) it takes for the stone to fall, then $d = 16t_1^2$, so $t_1 = \sqrt{d}/4$. Now if t_2 is the time it takes for the sound to travel back up, then $d = 1090t_2$ because the speed of sound is 1090 ft/s. So $t_2 = d/1090$. Thus, the total time elapsed between dropping the stone and hearing the splash is

$$t_1 + t_2 = \frac{\sqrt{d}}{4} + \frac{d}{1090}$$

How deep is the well if this total time is 3 s?



DISCOVERY ■ DISCUSSION ■ WRITING

- 120. A Family of Equations** The equation

$$3x + k - 5 = kx - k + 1$$

is really a **family of equations**, because for each value of k , we get a different equation with the unknown x . The letter k is called a **parameter** for this family. What value should we pick for k to make the given value of x a solution of the resulting equation?

- (a) $x = 0$ (b) $x = 1$ (c) $x = 2$

- 121. Proof That $0 = 1$** The following steps appear to give equivalent equations, which seem to prove that $1 = 0$. Find the error.

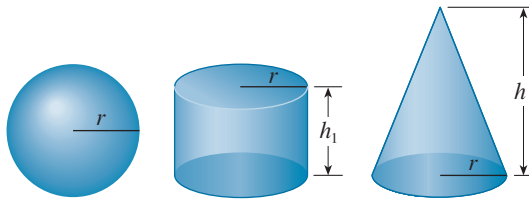
$x = 1$	Given
$x^2 = x$	Multiply by x
$x^2 - x = 0$	Subtract x
$x(x - 1) = 0$	Factor
$\frac{x(x - 1)}{x - 1} = \frac{0}{x - 1}$	Divide by $x - 1$
$x = 0$	Simplify
$1 = 0$	Given $x = 1$

- 122. Volumes of Solids** The sphere, cylinder, and cone shown here all have the same radius r and the same volume V .

(a) Use the volume formulas given on the inside front cover of this book, to show that

$$\frac{4}{3}\pi r^3 = \pi r^2 h_1 \quad \text{and} \quad \frac{4}{3}\pi r^3 = \frac{1}{3}\pi r^2 h_2$$

(b) Solve these equations for h_1 and h_2 .



123. Relationship Between Roots and Coefficients

The Quadratic Formula gives us the roots of a quadratic equation from its coefficients. We can also obtain the coefficients from the roots. For example, find the roots of the equation $x^2 - 9x + 20 = 0$ and show that the product of the roots is the constant term 20 and the sum of the roots is 9, the negative

of the coefficient of x . Show that the same relationship between roots and coefficients holds for the following equations:

$$x^2 - 2x - 8 = 0$$

$$x^2 + 4x + 2 = 0$$

Use the Quadratic Formula to prove that in general, if the equation $x^2 + bx + c = 0$ has roots r_1 and r_2 , then $c = r_1 r_2$ and $b = -(r_1 + r_2)$.

- 124. Solving an Equation in Different Ways** We have learned several different ways to solve an equation in this section. Some equations can be tackled by more than one method. For example, the equation $x - \sqrt{x} - 2 = 0$ is of quadratic type. We can solve it by letting $\sqrt{x} = u$ and $x = u^2$, and factoring. Or we could solve for $\sqrt{x}$, square each side, and then solve the resulting quadratic equation. Solve the following equations using both methods indicated, and show that you get the same final answers.

(a) $x - \sqrt{x} - 2 = 0$ quadratic type; solve for the radical, and square

(b) $\frac{12}{(x-3)^2} + \frac{10}{x-3} + 1 = 0$ quadratic type; multiply by LCD

1.6 MODELING WITH EQUATIONS

Making and Using Models ► Problems About Interest ► Problems About Area or Length ► Problems About Mixtures ► Problems About the Time Needed to Do a Job ► Problems About Distance, Rate, and Time

Many problems in the sciences, economics, finance, medicine, and numerous other fields can be translated into algebra problems; this is one reason that algebra is so useful. In this section we use equations as mathematical models to solve real-life problems.

▼ Making and Using Models

We will use the following guidelines to help us set up equations that model situations described in words. To show how the guidelines can help you to set up equations, we note them as we work each example in this section.

GUIDELINES FOR MODELING WITH EQUATIONS

- 1. Identify the Variable.** Identify the quantity that the problem asks you to find. This quantity can usually be determined by a careful reading of the question that is posed at the end of the problem. Then **introduce notation** for the variable (call it x or some other letter).
- 2. Translate from Words to Algebra.** Read each sentence in the problem again, and express all the quantities mentioned in the problem in terms of the variable you defined in Step 1. To organize this information, it is sometimes helpful to **draw a diagram** or **make a table**.
- 3. Set Up the Model.** Find the crucial fact in the problem that gives a relationship between the expressions you listed in Step 2. **Set up an equation** (or **model**) that expresses this relationship.
- 4. Solve the Equation and Check Your Answer.** Solve the equation, check your answer, and express it as a sentence that answers the question posed in the problem.

The following example illustrates how these guidelines are used to translate a “word problem” into the language of algebra.

EXAMPLE 1 | Renting a Car

A car rental company charges \$30 a day and 15¢ a mile for renting a car. Helen rents a car for two days, and her bill comes to \$108. How many miles did she drive?

SOLUTION

Identify the variable. We are asked to find the number of miles Helen has driven. So we let

$$x = \text{number of miles driven}$$

Translate from words to algebra. Now we translate all the information given in the problem into the language of algebra.

In Words	In Algebra
Number of miles driven	x
Mileage cost (at \$0.15 per mile)	$0.15x$
Daily cost (at \$30 per day)	$2(30)$

Set up the model. Now we set up the model.

$$\text{mileage cost} + \text{daily cost} = \text{total cost}$$

$$0.15x + 2(30) = 108$$

Solve. Now we solve for x .

$$0.15x = 48 \quad \text{Subtract 60}$$

$$x = \frac{48}{0.15} \quad \text{Divide by 0.15}$$

$$x = 320 \quad \text{Calculator}$$

CHECK YOUR ANSWER

$$\begin{aligned} \text{total cost} &= \text{mileage cost} + \text{daily cost} \\ &= 0.15(320) + 2(30) \\ &= 108 \quad \checkmark \end{aligned}$$

Helen drove her rental car 320 miles.

NOW TRY EXERCISE 19

In the examples and exercises that follow, we construct equations that model problems in many different real-life situations.

▼ Problems About Interest

When you borrow money from a bank or when a bank “borrows” your money by keeping it for you in a savings account, the borrower in each case must pay for the privilege of using the money. The fee that is paid is called **interest**. The most basic type of interest is **simple interest**, which is just an annual percentage of the total amount borrowed or deposited. The amount of a loan or deposit is called the **principal** P . The annual percentage paid for the use of this money is the **interest rate** r . We will use the variable t to stand for the number of years that the money is on deposit and the variable I to stand for the total interest earned. The following **simple interest formula** gives the amount of interest I earned when a principal P is deposited for t years at an interest rate r .

$$I = Prt$$

 When using this formula, remember to convert r from a percentage to a decimal. For example, in decimal form, 5% is 0.05. So at an interest rate of 5%, the interest paid on a \$1000 deposit over a 3-year period is $I = Prt = 1000(0.05)(3) = \150 .

EXAMPLE 2 | Interest on an Investment

Mary inherits \$100,000 and invests it in two certificates of deposit. One certificate pays 6% and the other pays $4\frac{1}{2}\%$ simple interest annually. If Mary's total interest is \$5025 per year, how much money is invested at each rate?

SOLUTION

Identify the variable. The problem asks for the amount she has invested at each rate. So we let

x = the amount invested at 6%

Translate from words to algebra. Since Mary's total inheritance is \$100,000, it follows that she invested $100,000 - x$ at $4\frac{1}{2}\%$. We translate all the information given into the language of algebra.

In Words	In Algebra
Amount invested at 6%	x
Amount invested at $4\frac{1}{2}\%$	$100,000 - x$
Interest earned at 6%	$0.06x$
Interest earned at $4\frac{1}{2}\%$	$0.045(100,000 - x)$

Set up the model. We use the fact that Mary's total interest is \$5025 to set up the model.

$$\text{interest at 6\%} + \text{interest at } 4\frac{1}{2}\% = \text{total interest}$$

$$0.06x + 0.045(100,000 - x) = 5025$$

Solve. Now we solve for x .

$$0.06x + 4500 - 0.045x = 5025 \quad \text{Distributive Property}$$

$$0.015x + 4500 = 5025 \quad \text{Combine the } x\text{-terms}$$

$$0.015x = 525 \quad \text{Subtract 4500}$$

$$x = \frac{525}{0.015} = 35,000 \quad \text{Divide by 0.015}$$

So Mary has invested \$35,000 at 6% and the remaining \$65,000 at $4\frac{1}{2}\%$.

CHECK YOUR ANSWER

$$\begin{aligned} \text{total interest} &= 6\% \text{ of } \$35,000 + 4\frac{1}{2}\% \text{ of } \$65,000 \\ &= \$2100 + \$2925 = \$5025 \quad \checkmark \end{aligned}$$

NOW TRY EXERCISE 21

▼ Problems About Area or Length

When we use algebra to model a physical situation, we must sometimes use basic formulas from geometry. For example, we may need a formula for an area or a perimeter, or the formula that relates the sides of similar triangles, or the Pythagorean Theorem. Most of these formulas are listed in the front endpapers of this book. The next two examples use these geometric formulas to solve some real-world problems.

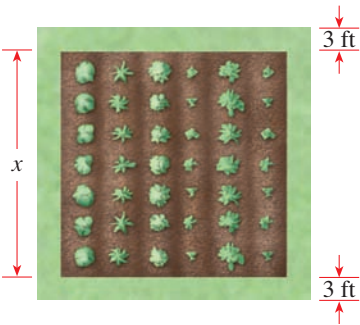


FIGURE 1

EXAMPLE 3 | Dimensions of a Garden

A square garden has a walkway 3 ft wide around its outer edge, as shown in Figure 1. If the area of the entire garden, including the walkway, is 18,000 ft², what are the dimensions of the planted area?

SOLUTION

Identify the variable. We are asked to find the length and width of the planted area. So we let

x = the length of the planted area

Translate from words to algebra. Next, translate the information from Figure 1 into the language of algebra.

In Words	In Algebra
Length of planted area	x
Length of entire garden	$x + 6$
Area of entire garden	$(x + 6)^2$

Set up the model. We now set up the model.

area of entire garden = 18,000 ft²

$$(x + 6)^2 = 18,000$$

Solve. Now we solve for x .

$$x + 6 = \sqrt{18,000}$$
$$x = \sqrt{18,000} - 6$$
$$x \approx 128$$

Take square roots
Subtract 6

The planted area of the garden is about 128 ft by 128 ft.

NOW TRY EXERCISE 47

EXAMPLE 4 | Dimensions of a Building Lot

A rectangular building lot is 8 ft longer than it is wide and has an area of 2900 ft². Find the dimensions of the lot.

SOLUTION

Identify the variable. We are asked to find the width and length of the lot. So let

w = width of lot

Translate from words to algebra. Then we translate the information given in the problem into the language of algebra (see Figure 2).

In Words	In Algebra
Width of lot	w
Length of Lot	$w + 8$

Set up the model. Now we set up the model.

width of lot

 ·

length of lot

 =

area of lot

$$w(w + 8) = 2900$$

Solve. Now we solve for w .

$$w^2 + 8w = 2900 \quad \text{Expand}$$

$$w^2 + 8w - 2900 = 0 \quad \text{Subtract 2900}$$

$$(w - 50)(w + 58) = 0 \quad \text{Factor}$$

$$w = 50 \quad \text{or} \quad w = -58 \quad \text{Zero-Product Property}$$

Since the width of the lot must be a positive number, we conclude that $w = 50$ ft. The length of the lot is $w + 8 = 50 + 8 = 58$ ft.

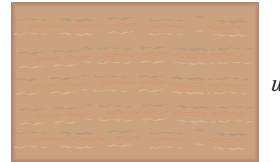


FIGURE 2

$w + 8$

 NOW TRY EXERCISE 39

EXAMPLE 5 | Determining the Height of a Building Using Similar Triangles

A man who is 6 ft tall wishes to find the height of a certain four-story building. He measures its shadow and finds it to be 28 ft long, while his own shadow is $3\frac{1}{2}$ ft long. How tall is the building?

SOLUTION

Identify the variable. The problem asks for the height of the building. So let

h = the height of the building

Translate from words to algebra. We use the fact that the triangles in Figure 3 are similar. Recall that for any pair of similar triangles the ratios of corresponding sides are equal. Now we translate these observations into the language of algebra.

In Words	In Algebra
Height of building	h
Ratio of height to base in large triangle	$\frac{h}{28}$
Ratio of height to base in small triangle	$\frac{6}{3\frac{1}{2}}$

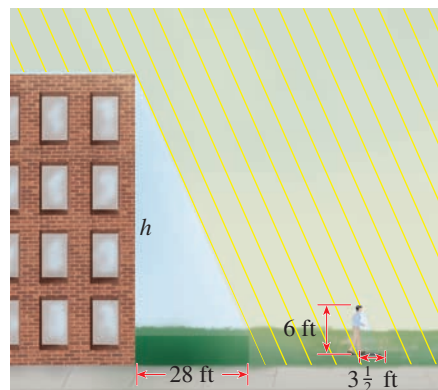


FIGURE 3

Set up the model. Since the large and small triangles are similar, we get the equation

ratio of height to
base in large triangle

=

ratio of height to
base in small triangle

$$\frac{h}{28} = \frac{6}{3.5}$$

Solve. Now we solve for h .

$$h = \frac{6 \cdot 28}{3.5} = 48 \quad \text{Multiply by 28}$$

So the building is 48 ft tall.

 **NOW TRY EXERCISE 51**

▼ **Problems About Mixtures**

Many real-world problems involve mixing different types of substances. For example, construction workers may mix cement, gravel, and sand; fruit juice from concentrate may involve mixing different types of juices. Problems involving mixtures and concentrations make use of the fact that if an amount x of a substance is dissolved in a solution with volume V , then the concentration C of the substance is given by

$$C = \frac{x}{V}$$

So if 10 g of sugar is dissolved in 5 L of water, then the sugar concentration is $C = 10/5 = 2$ g/L. Solving a mixture problem usually requires us to analyze the amount x of the substance that is in the solution. When we solve for x in this equation, we see that $x = CV$. Note that in many mixture problems the concentration C is expressed as a percentage, as in the next example.

EXAMPLE 6 | Mixtures and Concentration

A manufacturer of soft drinks advertises their orange soda as “naturally flavored,” although it contains only 5% orange juice. A new federal regulation stipulates that to be called “natural,” a drink must contain at least 10% fruit juice. How much pure orange juice must this manufacturer add to 900 gal of orange soda to conform to the new regulation?

SOLUTION

Identify the variable. The problem asks for the amount of pure orange juice to be added. So let

x = the amount (in gallons) of pure orange juice to be added

Translate from words to algebra. In any problem of this type—in which two different substances are to be mixed—drawing a diagram helps us to organize the given information (see Figure 4).

The information in the figure can be translated into the language of algebra, as follows:

In Words	In Algebra
Amount of orange juice to be added	x
Amount of the mixture	$900 + x$
Amount of orange juice in the first vat	$0.05(900) = 45$
Amount of orange juice in the second vat	$1 \cdot x = x$
Amount of orange juice in the mixture	$0.10(900 + x)$

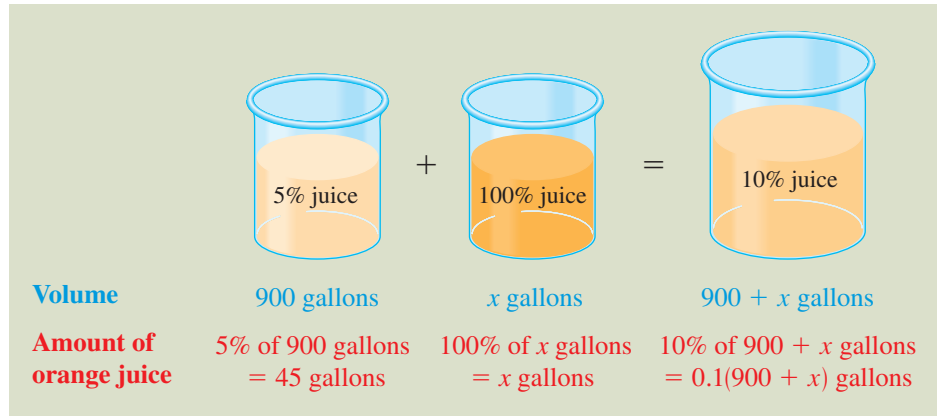


FIGURE 4

Set up the model. To set up the model, we use the fact that the total amount of orange juice in the mixture is equal to the orange juice in the first two vats.

amount of orange juice in first vat	+	amount of orange juice in second vat	=	amount of orange juice in mixture
---	---	--	---	---

$$45 + x = 0.1(900 + x) \quad \text{From Figure 4}$$

Solve. Now we solve for x .

$$45 + x = 90 + 0.1x \quad \text{Distributive Property}$$

$$0.9x = 45 \quad \text{Subtract } 0.1x \text{ and } 45$$

$$x = \frac{45}{0.9} = 50 \quad \text{Divide by } 0.9$$

The manufacturer should add 50 gal of pure orange juice to the soda.

CHECK YOUR ANSWER

$$\text{amount of juice before mixing} = 5\% \text{ of } 900 \text{ gal} + 50 \text{ gal pure juice}$$

$$= 45 \text{ gal} + 50 \text{ gal} = 95 \text{ gal}$$

$$\text{amount of juice after mixing} = 10\% \text{ of } 950 \text{ gal} = 95 \text{ gal}$$

Amounts are equal. ✓

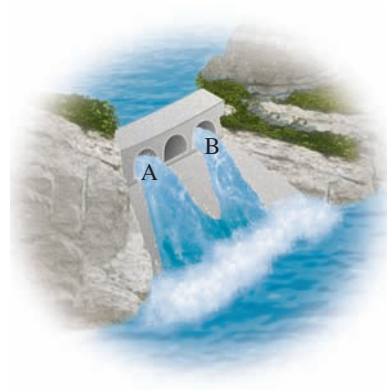
NOW TRY EXERCISE 53

▼ Problems About the Time Needed to Do a Job

When solving a problem that involves determining how long it takes several workers to complete a job, we use the fact that if a person or machine takes H time units to complete the task, then in one time unit the fraction of the task that has been completed is $1/H$. For example, if a worker takes 5 hours to mow a lawn, then in 1 hour the worker will mow $1/5$ of the lawn.

EXAMPLE 7 | Time Needed to Do a Job

Because of an anticipated heavy rainstorm, the water level in a reservoir must be lowered by 1 ft. Opening spillway A lowers the level by this amount in 4 hours, whereas opening the smaller spillway B does the job in 6 hours. How long will it take to lower the water level by 1 ft if both spillways are opened?



SOLUTION **Identify the variable.** We are asked to find the time needed to lower the level by 1 ft if both spillways are open. So let

x = the time (in hours) it takes to lower the water level by 1 ft if both spillways are open

Translate from words to algebra. Finding an equation relating x to the other quantities in this problem is not easy. Certainly x is not simply $4 + 6$, because that would mean that together the two spillways require longer to lower the water level than either spillway alone. Instead, we look at the fraction of the job that can be done in 1 hour by each spillway.

In Words	In Algebra
Time it takes to lower level 1 ft with A and B together	x h
Distance A lowers level in 1 h	$\frac{1}{4}$ ft
Distance B lowers level in 1 h	$\frac{1}{6}$ ft
Distance A and B together lower levels in 1 h	$\frac{1}{x}$ ft

Set up the model. Now we set up the model.

fraction done by A + fraction done by B = fraction done by both

$$\frac{1}{4} + \frac{1}{6} = \frac{1}{x}$$

Solve. Now we solve for x .

$$3x + 2x = 12 \quad \text{Multiply by the LCD, } 12x$$

$$5x = 12 \quad \text{Add}$$

$$x = \frac{12}{5} \quad \text{Divide by 5}$$

It will take $2\frac{2}{5}$ hours, or 2 h 24 min, to lower the water level by 1 ft if both spillways are open.

NOW TRY EXERCISE 61

▼ Problems About Distance, Rate, and Time

The next example deals with distance, rate (speed), and time. The formula to keep in mind here is

$$\text{distance} = \text{rate} \times \text{time}$$

where the rate is either the constant speed or average speed of a moving object. For example, driving at 60 mi/h for 4 hours takes you a distance of $60 \cdot 4 = 240$ mi.

EXAMPLE 8 | A Distance-Speed-Time Problem

A jet flew from New York to Los Angeles, a distance of 4200 km. The speed for the return trip was 100 km/h faster than the outbound speed. If the total trip took 13 hours, what was the jet's speed from New York to Los Angeles?

SOLUTION **Identify the variable.** We are asked for the speed of the jet from New York to Los Angeles. So let

s = speed from New York to Los Angeles

Then $s + 100$ = speed from Los Angeles to New York

Translate from words to algebra. Now we organize the information in a table. We fill in the “Distance” column first, since we know that the cities are 4200 km apart. Then we fill in the “Speed” column, since we have expressed both speeds (rates) in terms of the variable s . Finally, we calculate the entries for the “Time” column, using

$$\text{time} = \frac{\text{distance}}{\text{rate}}$$

	Distance (km)	Speed (km/h)	Time (h)
N.Y. to L.A.	4200	s	$\frac{4200}{s}$
L.A. to N.Y.	4200	$s + 100$	$\frac{4200}{s + 100}$

Set up the model. The total trip took 13 hours, so we have the model

$$\begin{array}{|c|} \hline \text{time from} \\ \text{N.Y. to L.A.} \\ \hline \end{array} + \begin{array}{|c|} \hline \text{time from} \\ \text{L.A. to N.Y.} \\ \hline \end{array} = \begin{array}{|c|} \hline \text{total} \\ \text{time} \\ \hline \end{array}$$

$$\frac{4200}{s} + \frac{4200}{s + 100} = 13$$

Solve. Multiplying by the common denominator, $s(s + 100)$, we get

$$4200(s + 100) + 4200s = 13s(s + 100)$$

$$8400s + 420,000 = 13s^2 + 1300s$$

$$0 = 13s^2 - 7100s - 420,000$$

Although this equation does factor, with numbers this large it is probably quicker to use the Quadratic Formula and a calculator.

$$\begin{aligned} s &= \frac{7100 \pm \sqrt{(-7100)^2 - 4(13)(-420,000)}}{2(13)} \\ &= \frac{7100 \pm 8500}{26} \end{aligned}$$

$$s = 600 \quad \text{or} \quad s = \frac{-1400}{26} \approx -53.8$$

Since s represents speed, we reject the negative answer and conclude that the jet’s speed from New York to Los Angeles was 600 km/h.

 **NOW TRY EXERCISE 67**

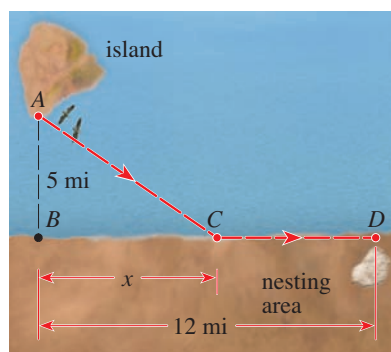


FIGURE 5

EXAMPLE 9 | Energy Expended in Bird Flight

Ornithologists have determined that some species of birds tend to avoid flights over large bodies of water during daylight hours, because air generally rises over land and falls over water in the daytime, so flying over water requires more energy. A bird is released from point A on an island, 5 mi from B, the nearest point on a straight shoreline. The bird flies to a point C on the shoreline and then flies along the shoreline to its nesting area D, as shown in Figure 5. Suppose the bird has 170 kcal of energy reserves. It uses 10 kcal/mi flying over land and 14 kcal/mi flying over water.

- Where should the point C be located so that the bird uses exactly 170 kcal of energy during its flight?
- Does the bird have enough energy reserves to fly directly from A to D?

Bhaskara (born 1114) was an Indian mathematician, astronomer, and astrologer. Among his many accomplishments was an ingenious proof of the Pythagorean Theorem. (See *Focus on Problem Solving* 5, Problem 12, at the book companion website www.stewartmath.com.) His important mathematical book *Lilavati* [*The Beautiful*] consists of algebra problems posed in the form of stories to his daughter Lilavati. Many of the problems begin “Oh beautiful maiden, suppose . . .” The story is told that using astrology, Bhaskara had determined that great misfortune would befall his daughter if she married at any time other than at a certain hour of a certain day. On her wedding day, as she was anxiously watching the water clock, a pearl fell unnoticed from her head-dress. It stopped the flow of water in the clock, causing her to miss the opportune moment for marriage. Bhaskara’s *Lilavati* was written to console her.

SOLUTION

- (a) **Identify the variable.** We are asked to find the location of C . So let

$$x = \text{distance from } B \text{ to } C$$

Translate from words to algebra. From the figure, and from the fact that

$$\text{energy used} = \text{energy per mile} \times \text{miles flown}$$

we determine the following:

In Words	In Algebra	
Distance from B to C	x	
Distance flown over water (from A to C)	$\sqrt{x^2 + 25}$	Pythagorean Theorem
Distance flown over land (from C to D)	$12 - x$	
Energy used over water	$14\sqrt{x^2 + 25}$	
Energy used over land	$10(12 - x)$	

Set up the model. Now we set up the model.

$$\begin{array}{|c|} \hline \text{total energy} \\ \text{used} \\ \hline \end{array} = \begin{array}{|c|} \hline \text{energy used} \\ \text{over water} \\ \hline \end{array} + \begin{array}{|c|} \hline \text{energy used} \\ \text{over land} \\ \hline \end{array}$$

$$170 = 14\sqrt{x^2 + 25} + 10(12 - x)$$

Solve. To solve this equation, we eliminate the square root by first bringing all other terms to the left of the equal sign and then squaring each side.

$$170 - 10(12 - x) = 14\sqrt{x^2 + 25} \quad \text{Isolate square-root term on RHS}$$

$$50 + 10x = 14\sqrt{x^2 + 25} \quad \text{Simplify LHS}$$

$$(50 + 10x)^2 = (14)^2(x^2 + 25) \quad \text{Square each side}$$

$$2500 + 1000x + 100x^2 = 196x^2 + 4900 \quad \text{Expand}$$

$$0 = 96x^2 - 1000x + 2400 \quad \text{All terms to RHS}$$

This equation could be factored, but because the numbers are so large it is easier to use the Quadratic Formula and a calculator:

$$\begin{aligned} x &= \frac{1000 \pm \sqrt{(-1000)^2 - 4(96)(2400)}}{2(96)} \\ &= \frac{1000 \pm 280}{192} = 6\frac{2}{3} \quad \text{or} \quad 3\frac{3}{4} \end{aligned}$$

Point C should be either $6\frac{2}{3}$ mi or $3\frac{3}{4}$ mi from B so that the bird uses exactly 170 kcal of energy during its flight.

- (b) By the Pythagorean Theorem (see page 219), the length of the route directly from A to D is $\sqrt{5^2 + 12^2} = 13$ mi, so the energy the bird requires for that route is $14 \times 13 = 182$ kcal. This is more energy than the bird has available, so it can't use this route.

 NOW TRY EXERCISE 83

1.6 EXERCISES

CONCEPTS

1. Explain in your own words what it means for an equation to model a real-world situation, and give an example.
2. In the formula $I = Prt$ for simple interest, P stands for _____, r for _____, and t for _____.
3. Give a formula for the area of the geometric figure.
 - (a) A square of side x : $A = \underline{\hspace{2cm}}$.
 - (b) A rectangle of length l and width w : $A = \underline{\hspace{2cm}}$.
 - (c) A circle of radius r : $A = \underline{\hspace{2cm}}$.
4. Balsamic vinegar contains 5% acetic acid, so a 32-oz bottle of balsamic vinegar contains _____ ounces of acetic acid.
5. A painter paints a wall in x hours, so the fraction of the wall that she paints in 1 hour is _____.
6. The formula $d = rt$ models the distance d traveled by an object moving at the constant rate r in time t . Find formulas for the following quantities.

$$r = \underline{\hspace{2cm}} \quad t = \underline{\hspace{2cm}}.$$

SKILLS

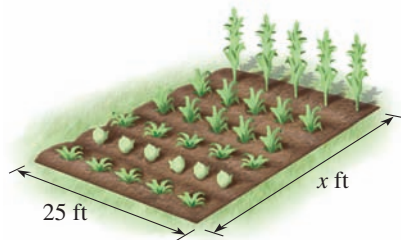
7–18 ■ Express the given quantity in terms of the indicated variable.

7. The sum of three consecutive integers; n = first integer of the three
8. The sum of three consecutive integers; n = middle integer of the three
9. The average of three test scores if the first two scores are 78 and 82; s = third test score
10. The average of four quiz scores if each of the first three scores is 8; q = fourth quiz score
11. The interest obtained after one year on an investment at $2\frac{1}{2}\%$ simple interest per year; x = number of dollars invested
12. The total rent paid for an apartment if the rent is \$795 a month; n = number of months
13. The area (in ft^2) of a rectangle that is three times as long as it is wide; w = width of the rectangle (in ft)
14. The perimeter (in cm) of a rectangle that is 5 cm longer than it is wide; w = width of the rectangle (in cm)
15. The distance (in mi) that a car travels in 45 min; s = speed of the car (in mi/h)
16. The time (in hours) it takes to travel a given distance at 55 mi/h; d = given distance (in mi)
17. The concentration (in oz/gal) of salt in a mixture of 3 gal of brine containing 25 oz of salt to which some pure water has been added; x = volume of pure water added (in gal)
18. The value (in cents) of the change in a purse that contains twice as many nickels as pennies, four more dimes than nickels, and as many quarters as dimes and nickels combined; p = number of pennies

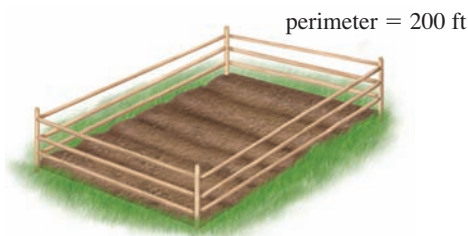
APPLICATIONS

19. **Renting a Truck** A rental company charges \$65 a day and 20 cents a mile for renting a truck. Michael rented a truck for 3 days, and his bill came to \$275. How many miles did he drive?
20. **Cell Phone Costs** A cell phone company charges a monthly fee of \$10 for the first 1000 text messages and 10 cents for each additional text message. Miriam's bill for text messages for the month of June is \$38.50. How many text messages did she send that month?
21. **Investments** Phyllis invested \$12,000, a portion earning a simple interest rate of $4\frac{1}{2}\%$ per year and the rest earning a rate of 4% per year. After 1 year the total interest earned on these investments was \$525. How much money did she invest at each rate?
22. **Investments** If Ben invests \$4000 at 4% interest per year, how much additional money must he invest at $5\frac{1}{2}\%$ annual interest to ensure that the interest he receives each year is $4\frac{1}{2}\%$ of the total amount invested?
23. **Investments** What annual rate of interest would you have to earn on an investment of \$3500 to ensure receiving \$262.50 interest after 1 year?
24. **Investments** Jack invests \$1000 at a certain annual interest rate, and he invests another \$2000 at an annual rate that is one-half percent higher. If he receives a total of \$190 interest in 1 year, at what rate is the \$1000 invested?
25. **Salaries** An executive in an engineering firm earns a monthly salary plus a Christmas bonus of \$8500. If she earns a total of \$97,300 per year, what is her monthly salary?
26. **Salaries** A woman earns 15% more than her husband. Together they make \$69,875 per year. What is the husband's annual salary?
27. **Inheritance** Craig is saving to buy a vacation home. He inherits some money from a wealthy uncle, then combines this with the \$22,000 he has already saved and doubles the total in a lucky investment. He ends up with \$134,000—just enough to buy a cabin on the lake. How much did he inherit?
28. **Overtime Pay** Helen earns \$7.50 an hour at her job, but if she works more than 35 hours in a week, she is paid $1\frac{1}{2}$ times her regular salary for the overtime hours worked. One week her gross pay was \$352.50. How many overtime hours did she work that week?
29. **Labor Costs** A plumber and his assistant work together to replace the pipes in an old house. The plumber charges \$45 an hour for his own labor and \$25 an hour for his assistant's labor. The plumber works twice as long as his assistant on this job, and the labor charge on the final bill is \$4025. How long did the plumber and his assistant work on this job?

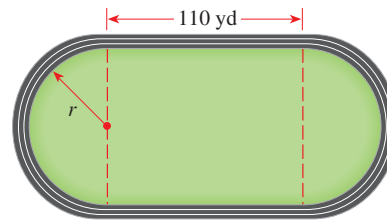
- 30. A Riddle** A father is four times as old as his daughter. In 6 years, he will be three times as old as she is. How old is the daughter now?
- 31. A Riddle** A movie star, unwilling to give his age, posed the following riddle to a gossip columnist: “Seven years ago, I was eleven times as old as my daughter. Now I am four times as old as she is.” How old is the movie star?
- 32. Career Home Runs** During his major league career, Hank Aaron hit 41 more home runs than Babe Ruth hit during his career. Together they hit 1469 home runs. How many home runs did Babe Ruth hit?
- 33. Value of Coins** A change purse contains an equal number of pennies, nickels, and dimes. The total value of the coins is \$1.44. How many coins of each type does the purse contain?
- 34. Value of Coins** Mary has \$3.00 in nickels, dimes, and quarters. If she has twice as many dimes as quarters and five more nickels than dimes, how many coins of each type does she have?
- 35. Length of a Garden** A rectangular garden is 25 ft wide. If its area is 1125 ft^2 , what is the length of the garden?



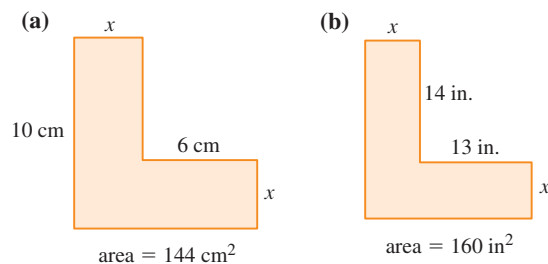
- 36. Width of a Pasture** A pasture is twice as long as it is wide. Its area is $115,200 \text{ ft}^2$. How wide is the pasture?
- 37. Dimensions of a Lot** A square plot of land has a building 60 ft long and 40 ft wide at one corner. The rest of the land outside the building forms a parking lot. If the parking lot has area $12,000 \text{ ft}^2$, what are the dimensions of the entire plot of land?
- 38. Dimensions of a Lot** A half-acre building lot is five times as long as it is wide. What are its dimensions? [Note: 1 acre = $43,560 \text{ ft}^2$.]
- 39. Dimensions of a Garden** A rectangular garden is 10 ft longer than it is wide. Its area is 875 ft^2 . What are its dimensions?
- 40. Dimensions of a Room** A rectangular bedroom is 7 ft longer than it is wide. Its area is 228 ft^2 . What is the width of the room?
- 41. Dimensions of a Garden** A farmer has a rectangular garden plot surrounded by 200 ft of fence. Find the length and width of the garden if its area is 2400 ft^2 .



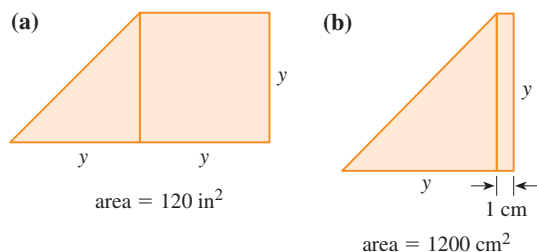
- 42. Dimensions of a Lot** A parcel of land is 6 ft longer than it is wide. Each diagonal from one corner to the opposite corner is 174 ft long. What are the dimensions of the parcel?
- 43. Dimensions of a Lot** A rectangular parcel of land is 50 ft wide. The length of a diagonal between opposite corners is 10 ft more than the length of the parcel. What is the length of the parcel?
- 44. Dimensions of a Track** A running track has the shape shown in the figure, with straight sides and semicircular ends. If the length of the track is 440 yd and the two straight parts are each 110 yd long, what is the radius of the semicircular parts (to the nearest yard)?



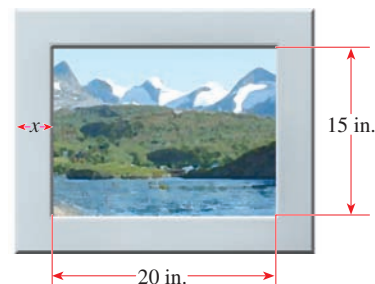
- 45. Length and Area** Find the length x in the figure. The area of the shaded region is given.



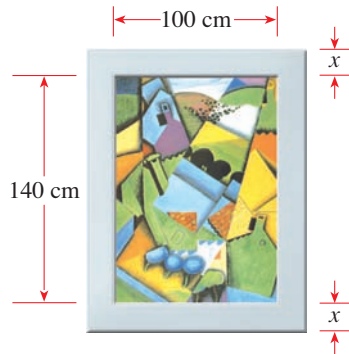
- 46. Length and Area** Find the length y in the figure. The area of the shaded region is given.



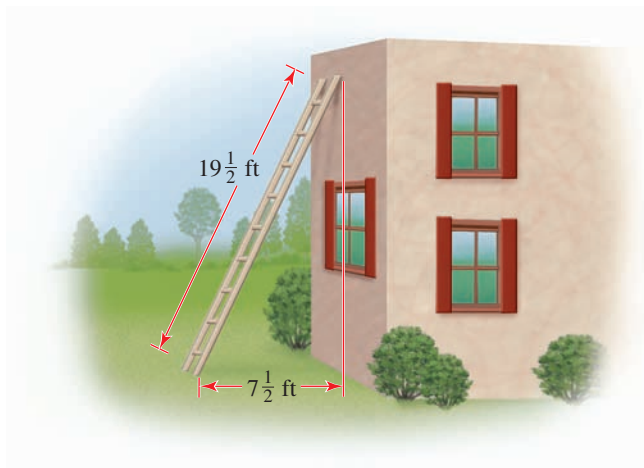
- 47. Framing a Painting** Ali paints with watercolors on a sheet of paper 20 in. wide by 15 in. high. He then places this sheet on a mat so that a uniformly wide strip of the mat shows all around the picture. The perimeter of the mat is 102 in. How wide is the strip of the mat showing around the picture?



- 48. Dimensions of a Poster** A poster has a rectangular printed area 100 cm by 140 cm and a blank strip of uniform width around the edges. The perimeter of the poster is $1\frac{1}{2}$ times the perimeter of the printed area. What is the width of the blank strip?



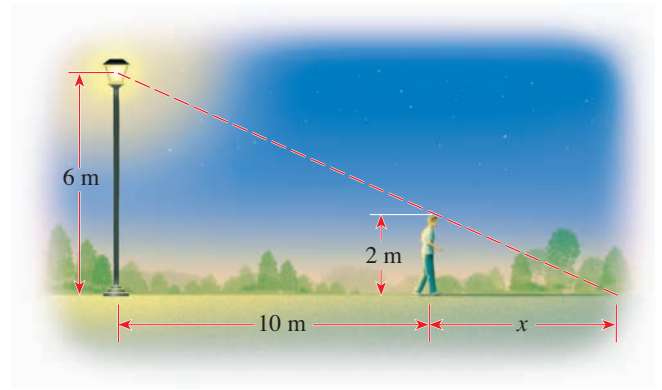
- 49. Reach of a Ladder** A $19\frac{1}{2}$ -foot ladder leans against a building. The base of the ladder is $7\frac{1}{2}$ ft from the building. How high up the building does the ladder reach?



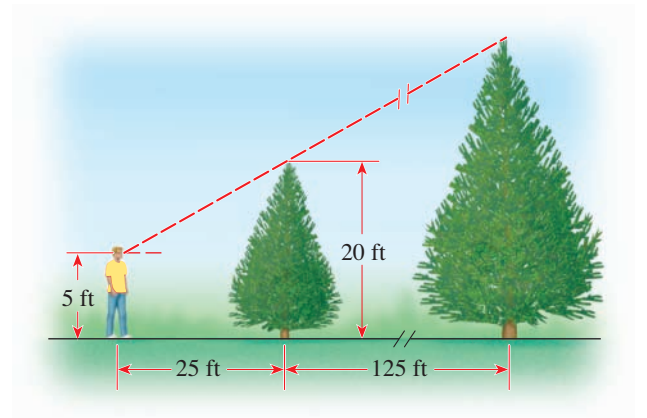
- 50. Height of a Flagpole** A flagpole is secured on opposite sides by two guy wires, each of which is 5 ft longer than the pole. The distance between the points where the wires are fixed to the ground is equal to the length of one guy wire. How tall is the flagpole (to the nearest inch)?



- 51. Length of a Shadow** A man is walking away from a lamppost with a light source 6 m above the ground. The man is 2 m tall. How long is the man's shadow when he is 10 m from the lamppost? [Hint: Use similar triangles.]



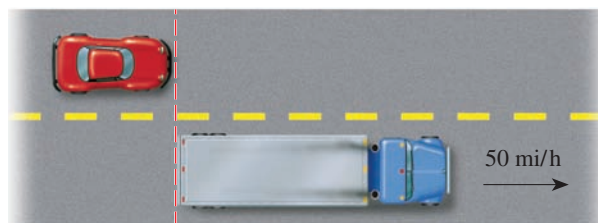
- 52. Height of a Tree** A woodcutter determines the height of a tall tree by first measuring a smaller one 125 ft away, then moving so that his eyes are in the line of sight along the tops of the trees and measuring how far he is standing from the small tree (see the figure). Suppose the small tree is 20 ft tall, the man is 25 ft from the small tree, and his eye level is 5 ft above the ground. How tall is the taller tree?



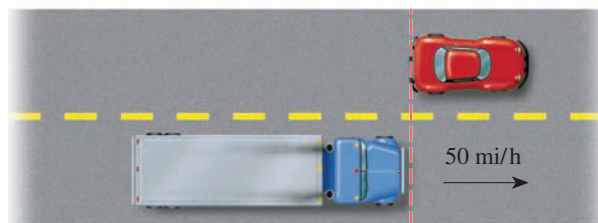
- 53. Mixture Problem** What quantity of a 60% acid solution must be mixed with a 30% solution to produce 300 mL of a 50% solution?
- 54. Mixture Problem** What quantity of pure acid must be added to 300 mL of a 50% acid solution to produce a 60% acid solution?
- 55. Mixture Problem** A jeweler has five rings, each weighing 18 g, made of an alloy of 10% silver and 90% gold. She decides to melt down the rings and add enough silver to reduce the gold content to 75%. How much silver should she add?
- 56. Mixture Problem** A pot contains 6 L of brine at a concentration of 120 g/L. How much of the water should be boiled off to increase the concentration to 200 g/L?
- 57. Mixture Problem** The radiator in a car is filled with a solution of 60% antifreeze and 40% water. The manufacturer of the antifreeze suggests that for summer driving, optimal cooling of the engine is obtained with only 50% antifreeze. If the capacity of the radiator is 3.6 L, how much coolant should be drained and replaced with water to reduce the antifreeze concentration to the recommended level?

- 58. Mixture Problem** A health clinic uses a solution of bleach to sterilize petri dishes in which cultures are grown. The sterilization tank contains 100 gal of a solution of 2% ordinary household bleach mixed with pure distilled water. New research indicates that the concentration of bleach should be 5% for complete sterilization. How much of the solution should be drained and replaced with bleach to increase the bleach content to the recommended level?
- 59. Mixture Problem** A bottle contains 750 mL of fruit punch with a concentration of 50% pure fruit juice. Jill drinks 100 mL of the punch and then refills the bottle with an equal amount of a cheaper brand of punch. If the concentration of juice in the bottle is now reduced to 48%, what was the concentration in the punch that Jill added?
- 60. Mixture Problem** A merchant blends tea that sells for \$3.00 a pound with tea that sells for \$2.75 a pound to produce 80 lb of a mixture that sells for \$2.90 a pound. How many pounds of each type of tea does the merchant use in the blend?
-  **61. Sharing a Job** Candy and Tim share a paper route. It takes Candy 70 min to deliver all the papers, and it takes Tim 80 min. How long does it take the two when they work together?
- 62. Sharing a Job** Stan and Hilda can mow the lawn in 40 min if they work together. If Hilda works twice as fast as Stan, how long does it take Stan to mow the lawn alone?
- 63. Sharing a Job** Betty and Karen have been hired to paint the houses in a new development. Working together, the women can paint a house in two-thirds the time that it takes Karen working alone. Betty takes 6 h to paint a house alone. How long does it take Karen to paint a house working alone?
- 64. Sharing a Job** Next-door neighbors Bob and Jim use hoses from both houses to fill Bob's swimming pool. They know that it takes 18 h using both hoses. They also know that Bob's hose, used alone, takes 20% less time than Jim's hose alone. How much time is required to fill the pool by each hose alone?
- 65. Sharing a Job** Henry and Irene working together can wash all the windows of their house in 1 h 48 min. Working alone, it takes Henry $1\frac{1}{2}$ h more than Irene to do the job. How long does it take each person working alone to wash all the windows?
- 66. Sharing a Job** Jack, Kay, and Lynn deliver advertising flyers in a small town. If each person works alone, it takes Jack 4 h to deliver all the flyers, and it takes Lynn 1 h longer than it takes Kay. Working together, they can deliver all the flyers in 40% of the time it takes Kay working alone. How long does it take Kay to deliver all the flyers alone?
-  **67. Distance, Speed, and Time** Wendy took a trip from Davenport to Omaha, a distance of 300 mi. She traveled part of the way by bus, which arrived at the train station just in time for Wendy to complete her journey by train. The bus averaged 40 mi/h, and the train averaged 60 mi/h. The entire trip took $5\frac{1}{2}$ h. How long did Wendy spend on the train?
- 68. Distance, Speed, and Time** Two cyclists, 90 mi apart, start riding toward each other at the same time. One cycles twice as fast as the other. If they meet 2 h later, at what average speed is each cyclist traveling?
- 69. Distance, Speed, and Time** A pilot flew a jet from Montreal to Los Angeles, a distance of 2500 mi. On the return trip, the average speed was 20% faster than the outbound speed. The round-trip took 9 h 10 min. What was the speed from Montreal to Los Angeles?

- 70. Distance, Speed, and Time** A woman driving a car 14 ft long is passing a truck 30 ft long. The truck is traveling at 50 mi/h. How fast must the woman drive her car so that she can pass the truck completely in 6 s, from the position shown in figure (a) to the position shown in figure (b)? [Hint: Use feet and seconds instead of miles and hours.]

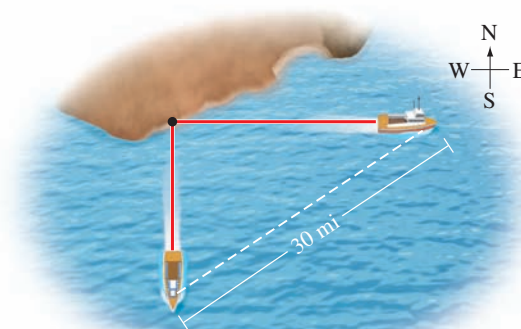


(a)



(b)

- 71. Distance, Speed, and Time** A salesman drives from Ajax to Barrington, a distance of 120 mi, at a steady speed. He then increases his speed by 10 mi/h to drive the 150 mi from Barrington to Collins. If the second leg of his trip took 6 min more time than the first leg, how fast was he driving between Ajax and Barrington?
- 72. Distance, Speed, and Time** Kiran drove from Tortula to Cactus, a distance of 250 mi. She increased her speed by 10 mi/h for the 360-mi trip from Cactus to Dry Junction. If the total trip took 11 h, what was her speed from Tortula to Cactus?
- 73. Distance, Speed, and Time** It took a crew 2 h 40 min to row 6 km upstream and back again. If the rate of flow of the stream was 3 km/h, what was the rowing speed of the crew in still water?
- 74. Speed of a Boat** Two fishing boats depart a harbor at the same time, one traveling east, the other south. The eastbound boat travels at a speed 3 mi/h faster than the southbound boat. After two hours the boats are 30 mi apart. Find the speed of the southbound boat.



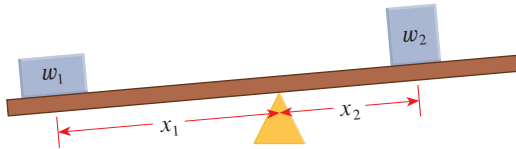
- 75. Law of the Lever** The figure shows a lever system, similar to a seesaw that you might find in a children's playground. For

the system to balance, the product of the weight and its distance from the fulcrum must be the same on each side; that is,

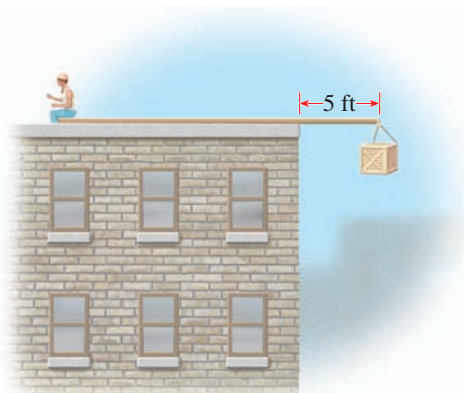
$$w_1x_1 = w_2x_2$$

This equation is called the **law of the lever** and was first discovered by Archimedes (see page 729).

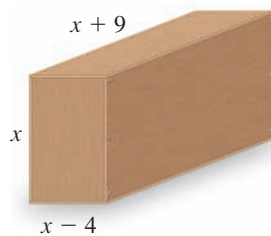
A woman and her son are playing on a seesaw. The boy is at one end, 8 ft from the fulcrum. If the son weighs 100 lb and the mother weighs 125 lb, where should the woman sit so that the seesaw is balanced?



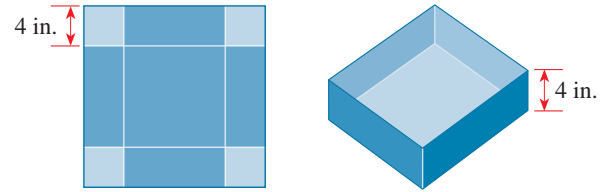
- 76. Law of the Lever** A plank 30 ft long rests on top of a flat-roofed building, with 5 ft of the plank projecting over the edge, as shown in the figure. A worker weighing 240 lb sits on one end of the plank. What is the largest weight that can be hung on the projecting end of the plank if it is to remain in balance? (Use the law of the lever stated in Exercise 75.)



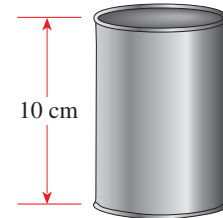
- 77. Dimensions of a Box** A large plywood box has a volume of 180 ft^3 . Its length is 9 ft greater than its height, and its width is 4 ft less than its height. What are the dimensions of the box?



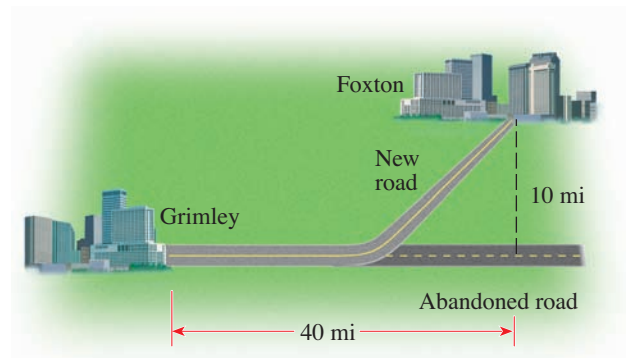
- 78. Radius of a Sphere** A jeweler has three small solid spheres made of gold, of radius 2 mm, 3 mm, and 4 mm. He decides to melt these down and make just one sphere out of them. What will the radius of this larger sphere be?
- 79. Dimensions of a Box** A box with a square base and no top is to be made from a square piece of cardboard by cutting 4-in. squares from each corner and folding up the sides, as shown in the figure. The box is to hold 100 in^3 . How big a piece of cardboard is needed?



- 80. Dimensions of a Can** A cylindrical can has a volume of $40\pi \text{ cm}^3$ and is 10 cm tall. What is its diameter? [Hint: Use the volume formula listed on the inside front cover of this book.]

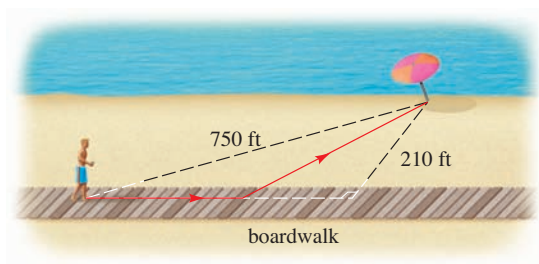


- 81. Radius of a Tank** A spherical tank has a capacity of 750 gallons. Using the fact that one gallon is about 0.1337 ft^3 , find the radius of the tank (to the nearest hundredth of a foot).
- 82. Dimensions of a Lot** A city lot has the shape of a right triangle whose hypotenuse is 7 ft longer than one of the other sides. The perimeter of the lot is 392 ft. How long is each side of the lot?
- 83. Construction Costs** The town of Foxton lies 10 mi north of an abandoned east-west road that runs through Grimley, as shown in the figure. The point on the abandoned road closest to Foxton is 40 mi from Grimley. County officials are about to build a new road connecting the two towns. They have determined that restoring the old road would cost \$100,000 per mile, whereas building a new road would cost \$200,000 per mile. How much of the abandoned road should be used (as indicated in the figure) if the officials intend to spend exactly \$6.8 million? Would it cost less than this amount to build a new road connecting the towns directly?

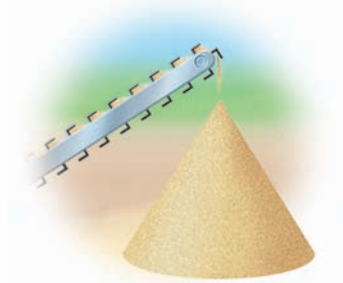


- 84. Distance, Speed, and Time** A boardwalk is parallel to and 210 ft inland from a straight shoreline. A sandy beach lies between the boardwalk and the shoreline. A man is standing on the boardwalk, exactly 750 ft across the sand from his beach umbrella, which is right at the shoreline. The man walks 4 ft/s on the boardwalk and 2 ft/s on the sand. How far should

he walk on the boardwalk before veering off onto the sand if he wishes to reach his umbrella in exactly 4 min 45 s?



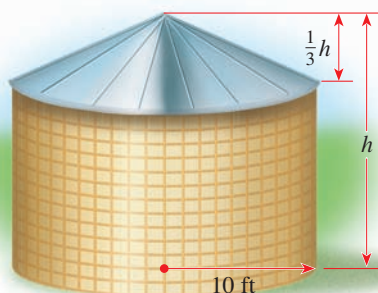
- 85. Volume of Grain** Grain is falling from a chute onto the ground, forming a conical pile whose diameter is always three times its height. How high is the pile (to the nearest hundredth of a foot) when it contains 1000 ft^3 of grain?



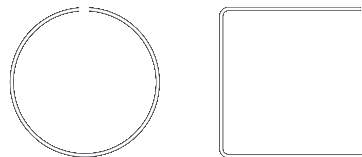
- 86. TV Monitors** Two television monitors sitting side by side on a shelf in an appliance store have the same screen height. One has a conventional screen, which is 5 in. wider than it is high. The other has a wider, high-definition screen, which is 1.8 times as wide as it is high. The diagonal measure of the wider screen is 14 in. more than the diagonal measure of the smaller screen. What is the height of the screens, correct to the nearest 0.1 in.?



- 87. Dimensions of a Structure** A storage bin for corn consists of a cylindrical section made of wire mesh, surmounted by a conical tin roof, as shown in the figure. The height of the roof is one-third the height of the entire structure. If the total volume of the structure is $1400\pi \text{ ft}^3$ and its radius is 10 ft, what is its height? [Hint: Use the volume formulas listed on the inside front cover of this book.]



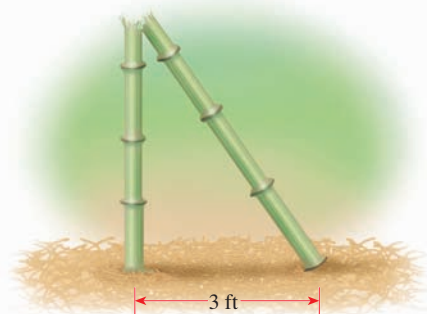
- 88. Comparing Areas** A wire 360 in. long is cut into two pieces. One piece is formed into a square, and the other is formed into a circle. If the two figures have the same area, what are the lengths of the two pieces of wire (to the nearest tenth of an inch)?



- 89. An Ancient Chinese Problem** This problem is taken from a Chinese mathematics textbook called *Chui-chang suan-shu*, or *Nine Chapters on the Mathematical Art*, which was written about 250 B.C.

A 10-ft-long stem of bamboo is broken in such a way that its tip touches the ground 3 ft from the base of the stem, as shown in the figure. What is the height of the break?

[Hint: Use the Pythagorean Theorem.]



DISCOVERY ■ DISCUSSION ■ WRITING

- 90. Historical Research** Read the biographical notes on Pythagoras (page 219), Euclid (page 497), and Archimedes (page 729). Choose one of these mathematicians and find out more about him from the library or on the Internet. Write a short essay on your findings. Include both biographical information and a description of the mathematics for which he is famous.

- 91. A Babylonian Quadratic Equation** The ancient Babylonians knew how to solve quadratic equations. Here is a problem from a cuneiform tablet found in a Babylonian school dating back to about 2000 B.C.

I have a reed, I know not its length. I broke from it one cubit, and it fit 60 times along the length of my field. I re-stored to the reed what I had broken off, and it fit 30 times along the width of my field. The area of my field is 375 square nindas. What was the original length of the reed?

Solve this problem. Use the fact that 1 ninda = 12 cubits.



DISCOVERY PROJECT

Equations Through the Ages

In this project we study equations that were created and solved by the ancient peoples of Egypt, Babylon, India, and China. You can find the project at the book companion website: www.stewartmath.com

1.7 INEQUALITIES

Solving Linear Inequalities ► Solving Nonlinear Inequalities ► Absolute Value Inequalities ► Modeling with Inequalities

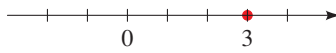
Some problems in algebra lead to **inequalities** instead of equations. An inequality looks just like an equation, except that in the place of the equal sign is one of the symbols, $<$, $>$, $\leq$, or $\geq$. Here is an example of an inequality:

$$4x + 7 \leq 19$$

x	$4x + 7 \leq 19$
1	$11 \leq 19$ ✓
2	$15 \leq 19$ ✓
3	$19 \leq 19$ ✓
4	$23 \leq 19$ ✗
5	$27 \leq 19$ ✗

The table in the margin shows that some numbers satisfy the inequality and some numbers don't.

To **solve** an inequality that contains a variable means to find all values of the variable that make the inequality true. Unlike an equation, an inequality generally has infinitely many solutions, which form an interval or a union of intervals on the real line. The following illustration shows how an inequality differs from its corresponding equation:

	Solution	Graph
Equation: $4x + 7 = 19$	$x = 3$	
Inequality: $4x + 7 \leq 19$	$x \leq 3$	

To solve inequalities, we use the following rules to isolate the variable on one side of the inequality sign. These rules tell us when two inequalities are *equivalent* (the symbol $\Leftrightarrow$ means “is equivalent to”). In these rules the symbols A , B , and C stand for real numbers or algebraic expressions. Here we state the rules for inequalities involving the symbol $\leq$, but they apply to all four inequality symbols.

RULES FOR INEQUALITIES

Rule

- $A \leq B \Leftrightarrow A + C \leq B + C$
- $A \leq B \Leftrightarrow A - C \leq B - C$
- If $C > 0$, then $A \leq B \Leftrightarrow CA \leq CB$
- If $C < 0$, then $A \leq B \Leftrightarrow CA \geq CB$
- If $A > 0$ and $B > 0$,
then $A \leq B \Leftrightarrow \frac{1}{A} \geq \frac{1}{B}$
- If $A \leq B$ and $C \leq D$,
then $A + C \leq B + D$

Description

Adding the same quantity to each side of an inequality gives an equivalent inequality.

Subtracting the same quantity from each side of an inequality gives an equivalent inequality.

Multiplying each side of an inequality by the same *positive* quantity gives an equivalent inequality.

Multiplying each side of an inequality by the same *negative* quantity *reverses the direction* of the inequality.

Taking reciprocals of each side of an inequality involving *positive* quantities *reverses the direction* of the inequality.

Inequalities can be added.



Pay special attention to Rules 3 and 4. Rule 3 says that we can multiply (or divide) each side of an inequality by a *positive* number, but Rule 4 says that **if we multiply each side of an inequality by a *negative* number, then we reverse the direction of the inequality.** For example, if we start with the inequality

$$3 < 5$$

and multiply by 2, we get

$$6 < 10$$

but if we multiply by -2 , we get

$$-6 > -10$$

▼ Solving Linear Inequalities

An inequality is **linear** if each term is constant or a multiple of the variable. To solve a linear inequality, we isolate the variable on one side of the inequality sign.

EXAMPLE 1 | Solving a Linear Inequality

Solve the inequality $3x < 9x + 4$ and sketch the solution set.

SOLUTION

$$3x < 9x + 4 \quad \text{Given inequality}$$

$$3x - 9x < 9x + 4 - 9x \quad \text{Subtract } 9x$$

$$-6x < 4 \quad \text{Simplify}$$

$$\left(-\frac{1}{6}\right)(-6x) > \left(-\frac{1}{6}\right)(4) \quad \text{Multiply by } -\frac{1}{6} \text{ and reverse inequality}$$

$$x > -\frac{2}{3} \quad \text{Simplify}$$

Multiplying by the negative number $-\frac{1}{6}$ reverses the direction of the inequality.



FIGURE 1

The solution set consists of all numbers greater than $-\frac{2}{3}$. In other words the solution of the inequality is the interval $(-\frac{2}{3}, \infty)$. It is graphed in Figure 1.

NOW TRY EXERCISE 21

EXAMPLE 2 | Solving a Pair of Simultaneous Inequalities

Solve the inequalities $4 \leq 3x - 2 < 13$.

SOLUTION The solution set consists of all values of x that satisfy both of the inequalities $4 \leq 3x - 2$ and $3x - 2 < 13$. Using Rules 1 and 3, we see that the following inequalities are equivalent:

$$4 \leq 3x - 2 < 13 \quad \text{Given inequality}$$

$$6 \leq 3x < 15 \quad \text{Add 2}$$

$$2 \leq x < 5 \quad \text{Divide by 3}$$

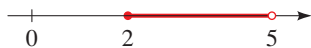


FIGURE 2

Therefore, the solution set is $[2, 5)$, as shown in Figure 2.

NOW TRY EXERCISE 31

▼ Solving Nonlinear Inequalities

To solve inequalities involving squares and other powers of the variable, we use factoring, together with the following principle.

THE SIGN OF A PRODUCT OR QUOTIENT

If a product or a quotient has an *even* number of *negative* factors, then its value is *positive*.

If a product or a quotient has an *odd* number of *negative* factors, then its value is *negative*.

For example, to solve the inequality $x^2 - 5x \leq -6$, we first move all terms to the left-hand side and factor to get

$$(x - 2)(x - 3) \leq 0$$

This form of the inequality says that the product $(x - 2)(x - 3)$ must be negative or zero, so to solve the inequality, we must determine where each factor is negative or positive (because the sign of a product depends on the sign of the factors). The details are explained in Example 3, in which we use the following guidelines.

GUIDELINES FOR SOLVING NONLINEAR INEQUALITIES

- 1. Move All Terms to One Side.** If necessary, rewrite the inequality so that all nonzero terms appear on one side of the inequality sign. If the nonzero side of the inequality involves quotients, bring them to a common denominator.
- 2. Factor.** Factor the nonzero side of the inequality.
- 3. Find the Intervals.** Determine the values for which each factor is zero. These numbers will divide the real line into intervals. List the intervals that are determined by these numbers.
- 4. Make a Table or Diagram.** Use test values to make a table or diagram of the signs of each factor on each interval. In the last row of the table determine the sign of the product (or quotient) of these factors.
- 5. Solve.** Determine the solution of the inequality from the last row of the sign table. Be sure to check whether the inequality is satisfied by some or all of the endpoints of the intervals. (This may happen if the inequality involves $\leq$ or $\geq$.)



The factoring technique that is described in these guidelines works only if all nonzero terms appear on one side of the inequality symbol. If the inequality is not written in this form, first rewrite it, as indicated in Step 1.

EXAMPLE 3 | Solving a Quadratic Inequality

Solve the inequality $x^2 \leq 5x - 6$.

SOLUTION We will follow the guidelines given above.

Move all terms to one side. We move all the terms to the left-hand side.

$$\begin{aligned} x^2 &\leq 5x - 6 && \text{Given inequality} \\ x^2 - 5x + 6 &\leq 0 && \text{Subtract } 5x, \text{ add } 6 \end{aligned}$$

Factor. Factoring the left-hand side of the inequality, we get

$$(x - 2)(x - 3) \leq 0 \quad \text{Factor}$$

Find the intervals. The factors of the left-hand side are $x - 2$ and $x - 3$. These factors are zero when x is 2 and 3, respectively. As shown in Figure 3, the numbers 2 and 3 divide the real line into the three intervals

$$(-\infty, 2), (2, 3), (3, \infty)$$

The factors $x - 2$ and $x - 3$ change sign only at 2 and 3, respectively. So these factors maintain their sign on each of these three intervals.

Make a table or diagram. To determine the sign of each factor on each of the intervals that we found, we use **test values**. We choose a number inside each interval and check the sign of the factors $x - 2$ and $x - 3$ at the number we chose. For the interval $(-\infty, 2)$, let's choose the test value 1 (see Figure 4). Substituting 1 for x in the factors $x - 2$ and $x - 3$, we get

$$\begin{aligned} x - 2 &= 1 - 2 = -1 < 0 \\ x - 3 &= 1 - 3 = -2 < 0 \end{aligned}$$

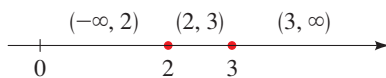


FIGURE 3

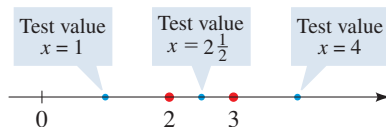


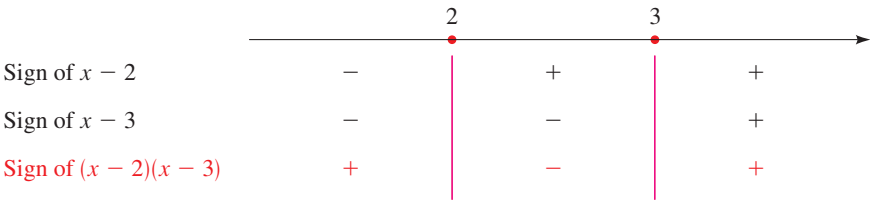
FIGURE 4

So both factors are negative on this interval. Notice that we need to check only one test value for each interval because the factors $x - 2$ and $x - 3$ do not change sign on any of the three intervals we found.

Using the test values $x = 2\frac{1}{2}$ and $x = 4$ for the intervals $(2, 3)$ and $(3, \infty)$ (see Figure 4), respectively, we construct the following sign table. The final row of the table is obtained from the fact that the expression in the last row is the product of the two factors.

Interval	$(-\infty, 2)$	$(2, 3)$	$(3, \infty)$
Sign of $x - 2$	−	+	+
Sign of $x - 3$	−	−	+
Sign of $(x - 2)(x - 3)$	+	−	+

If you prefer, you can represent this information on a real line, as in the following sign diagram. The vertical lines indicate the points at which the real line is divided into intervals:



Solve. We read from the table or the diagram that $(x - 2)(x - 3)$ is negative on the interval $(2, 3)$. Thus, the solution of the inequality $(x - 2)(x - 3) \leq 0$ is

$$\{x \mid 2 \leq x \leq 3\} = [2, 3]$$

We have included the endpoints 2 and 3 because we seek values of x such that the product is either less than *or equal to* zero. The solution is illustrated in Figure 5.

 NOW TRY EXERCISE 41

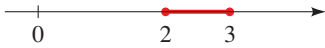


FIGURE 5

EXAMPLE 4 | Solving an Inequality with Repeated Factors

Solve the inequality $x(x - 1)^2(x - 3) < 0$.

SOLUTION All nonzero terms are already on one side of the inequality, and the nonzero side of the inequality is already factored. So we begin by finding the intervals for this inequality.

Find the intervals. The factors of the left-hand side are x , $(x - 1)^2$, and $x - 3$. These are zero when $x = 0, 1, 3$. These numbers divide the real line into the intervals

$$(-\infty, 0), (0, 1), (1, 3), (3, \infty)$$

Make a diagram. We make the following diagram, using test points to determine the sign of each factor in each interval.

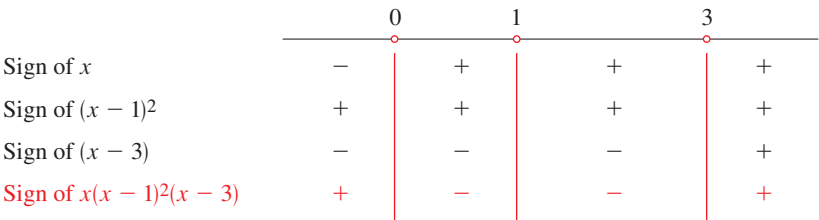




FIGURE 6

Solve. From the diagram we see that $x(x - 1)^2(x - 3) < 0$ for x in the interval $(0, 1)$ or for x in $(1, 3)$. So the solution set is the union of these two intervals:

$$(0, 1) \cup (1, 3)$$

The solution set is graphed in Figure 6.

NOW TRY EXERCISE 53

EXAMPLE 5 | Solving an Inequality Involving a Quotient

Solve the inequality $\frac{1 + x}{1 - x} \geq 1$.

SOLUTION

Move all terms to one side. We move the terms to the left-hand side and simplify using a common denominator.

$$\frac{1 + x}{1 - x} \geq 1 \quad \text{Given inequality}$$

$$\frac{1 + x}{1 - x} - 1 \geq 0 \quad \text{Subtract 1}$$

$$\frac{1 + x}{1 - x} - \frac{1 - x}{1 - x} \geq 0 \quad \text{Common denominator } 1 - x$$

$$\frac{1 + x - 1 + x}{1 - x} \geq 0 \quad \text{Combine the fractions}$$

$$\frac{2x}{1 - x} \geq 0 \quad \text{Simplify}$$

It is tempting to simply multiply both sides of the inequality by $1 - x$ (as you would if this were an *equation*). But this doesn't work because we don't know whether $1 - x$ is positive or negative, so we can't tell whether the inequality needs to be reversed. (See Exercise 123.)

Find the intervals. The factors of the left-hand side are $2x$ and $1 - x$. These are zero when x is 0 and 1. These numbers divide the real line into the intervals

$$(-\infty, 0), (0, 1), (1, \infty)$$

Make a diagram. We make the following diagram using test points to determine the sign of each factor in each interval.

		0		1	
Sign of $2x$	−		+		+
Sign of $1 - x$	+		+		−
Sign of $\frac{2x}{1 - x}$	−		+		−

Solve. From the diagram we see that $\frac{2x}{1 - x} \geq 0$ for x in the interval $[0, 1)$. We include the endpoint 0 because the original inequality requires that the quotient be greater than or equal to 1. However, we do not include the other endpoint 1 because the quotient in the inequality is not defined at 1. So the solution set is the interval

$$[0, 1)$$

The solution set is graphed in Figure 7.

NOW TRY EXERCISE 59



FIGURE 7



Example 5 shows that we should **always check the endpoints of the solution set to see whether they satisfy the original inequality.**

▼ Absolute Value Inequalities

We use the following properties to solve inequalities that involve absolute value.

These properties hold when x is replaced by any algebraic expression. (In the figures we assume that $c > 0$.)

PROPERTIES OF ABSOLUTE VALUE INEQUALITIES

Inequality	Equivalent form	Graph
1. $ x < c$	$-c < x < c$	
2. $ x \leq c$	$-c \leq x \leq c$	
3. $ x > c$	$x < -c$ or $c < x$	
4. $ x \geq c$	$x \leq -c$ or $c \leq x$	

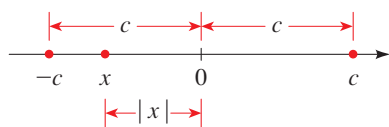


FIGURE 8

These properties can be proved using the definition of absolute value. To prove Property 1, for example, note that the inequality $|x| < c$ says that the distance from x to 0 is less than c , and from Figure 8 you can see that this is true if and only if x is between $-c$ and c .

EXAMPLE 6 | Solving an Absolute Value Inequality

Solve the inequality $|x - 5| < 2$.

SOLUTION 1 The inequality $|x - 5| < 2$ is equivalent to

$$-2 < x - 5 < 2 \quad \text{Property 1}$$

$$3 < x < 7 \quad \text{Add 5}$$

The solution set is the open interval $(3, 7)$.

SOLUTION 2 Geometrically, the solution set consists of all numbers x whose distance from 5 is less than 2. From Figure 9 we see that this is the interval $(3, 7)$.

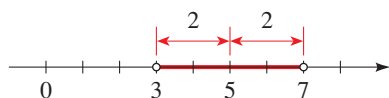


FIGURE 9

NOW TRY EXERCISE 79

EXAMPLE 7 | Solving an Absolute Value Inequality

Solve the inequality $|3x + 2| \geq 4$.

SOLUTION By Property 4 the inequality $|3x + 2| \geq 4$ is equivalent to

$$3x + 2 \geq 4 \quad \text{or} \quad 3x + 2 \leq -4$$

$$3x \geq 2 \quad \quad \quad 3x \leq -6 \quad \text{Subtract 2}$$

$$x \geq \frac{2}{3} \quad \quad \quad x \leq -2 \quad \text{Divide by 3}$$

So the solution set is

$$\{x \mid x \leq -2 \text{ or } x \geq \frac{2}{3}\} = (-\infty, -2] \cup [\frac{2}{3}, \infty)$$

The set is graphed in Figure 10.

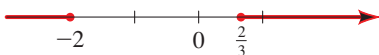


FIGURE 10

NOW TRY EXERCISE 83

▼ Modeling with Inequalities

Modeling real-life problems frequently leads to inequalities because we are often interested in determining when one quantity is more (or less) than another.

EXAMPLE 8 | Carnival Tickets

A carnival has two plans for tickets.

Plan A: \$5 entrance fee and 25¢ each ride

Plan B: \$2 entrance fee and 50¢ each ride

How many rides would you have to take for Plan A to be less expensive than Plan B?

SOLUTION **Identify the variable.** We are asked for the number of rides for which Plan A is less expensive than Plan B. So let

$$x = \text{number of rides}$$

Translate from words to algebra. The information in the problem may be organized as follows.

In Words	In Algebra
Number of rides	x
Cost with Plan A	$5 + 0.25x$
Cost with Plan B	$2 + 0.50x$

Set up the model. Now we set up the model.

$$\begin{array}{|c|} \hline \text{cost with} \\ \text{Plan A} \\ \hline \end{array} < \begin{array}{|c|} \hline \text{cost with} \\ \text{Plan B} \\ \hline \end{array}$$

$$5 + 0.25x < 2 + 0.50x$$

Solve. Now we solve for x .

$$3 + 0.25x < 0.50x \quad \text{Subtract 2}$$

$$3 < 0.25x \quad \text{Subtract } 0.25x$$

$$12 < x \quad \text{Divide by } 0.25$$

So if you plan to take *more than* 12 rides, Plan A is less expensive.

 **NOW TRY EXERCISE 107**

EXAMPLE 9 | Relationship Between Fahrenheit and Celsius Scales

The instructions on a bottle of medicine indicate that the bottle should be stored at a temperature between 5°C and 30°C . What range of temperatures does this correspond to on the Fahrenheit scale?

SOLUTION The relationship between degrees Celsius (C) and degrees Fahrenheit (F) is given by the equation $C = \frac{5}{9}(F - 32)$. Expressing the statement on the bottle in terms of inequalities, we have

$$5 < C < 30$$

So the corresponding Fahrenheit temperatures satisfy the inequalities

$$5 < \frac{5}{9}(F - 32) < 30 \quad \text{Substitute } C = \frac{5}{9}(F - 32)$$

$$\frac{9}{5} \cdot 5 < F - 32 < \frac{9}{5} \cdot 30 \quad \text{Multiply by } \frac{9}{5}$$

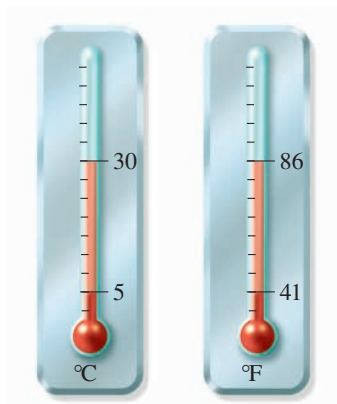
$$9 < F - 32 < 54 \quad \text{Simplify}$$

$$9 + 32 < F < 54 + 32 \quad \text{Add 32}$$

$$41 < F < 86 \quad \text{Simplify}$$

The medicine should be stored at a temperature between 41°F and 86°F .

 **NOW TRY EXERCISE 105**



1.7 EXERCISES

CONCEPTS

- Fill in the blank with an appropriate inequality sign.
 - If $x < 5$, then $x - 3$ _____ 2.
 - If $x \leq 5$, then $3x$ _____ 15.
 - If $x \geq 2$, then $-3x$ _____ -6.
 - If $x < -2$, then $-x$ _____ 2.
- True or false?
 - If $x(x + 1) > 0$, then x and $x + 1$ are either both positive or both negative.
 - If $x(x + 1) > 5$, then x and $x + 1$ are each greater than 5.
- (a) The solution of the inequality $|x| \leq 3$ is the interval _____.
- (b) The solution of the inequality $|x| \geq 3$ is a union of two intervals _____ $\cup$ _____.
- (a) The set of all points on the real line whose distance from zero is less than 3 can be described by the absolute value inequality $|x|$ _____.
- (b) The set of all points on the real line whose distance from zero is greater than 3 can be described by the absolute value inequality $|x|$ _____.

SKILLS

5–10 ■ Let $S = \{-2, -1, 0, \frac{1}{2}, 1, \sqrt{2}, 2, 4\}$. Determine which elements of S satisfy the inequality.

- $3 - 2x \leq \frac{1}{2}$
- $2x - 1 \geq x$
- $1 < 2x - 4 \leq 7$
- $-2 \leq 3 - x < 2$
- $\frac{1}{x} \leq \frac{1}{2}$
- $x^2 + 2 < 4$

11–34 ■ Solve the linear inequality. Express the solution using interval notation and graph the solution set.

- $2x \leq 7$
- $-4x \geq 10$
- $2x - 5 > 3$
- $3x + 11 < 5$
- $7 - x \geq 5$
- $5 - 3x \leq -16$
- $2x + 1 < 0$
- $0 < 5 - 2x$
- $3x + 11 \leq 6x + 8$
- $6 - x \geq 2x + 9$
- $\frac{1}{2}x - \frac{2}{3} > 2$
- $\frac{2}{5}x + 1 < \frac{1}{5} - 2x$
- $\frac{1}{3}x + 2 < \frac{1}{6}x - 1$
- $\frac{2}{3} - \frac{1}{2}x \geq \frac{1}{6} + x$
- $4 - 3x \leq -(1 + 8x)$
- $2(7x - 3) \leq 12x + 16$
- $2 \leq x + 5 < 4$
- $5 \leq 3x - 4 \leq 14$
- $-1 < 2x - 5 < 7$
- $1 < 3x + 4 \leq 16$
- $-2 < 8 - 2x \leq -1$
- $-3 \leq 3x + 7 \leq \frac{1}{2}$
- $\frac{1}{6} < \frac{2x - 13}{12} \leq \frac{2}{3}$
- $-\frac{1}{2} \leq \frac{4 - 3x}{5} \leq \frac{1}{4}$

35–72 ■ Solve the nonlinear inequality. Express the solution using interval notation and graph the solution set.

- $(x + 2)(x - 3) < 0$
- $(x - 5)(x + 4) \geq 0$
- $x(2x + 7) \geq 0$
- $x(2 - 3x) \leq 0$
- $x^2 - 3x - 18 \leq 0$
- $x^2 + 5x + 6 > 0$
- $2x^2 + x \geq 1$
- $x^2 < x + 2$
- $3x^2 - 3x < 2x^2 + 4$
- $5x^2 + 3x \geq 3x^2 + 2$
- $x^2 > 3(x + 6)$
- $x^2 + 2x > 3$
- $x^2 < 4$
- $x^2 \geq 9$
- $(x + 2)(x - 1)(x - 3) \leq 0$
- $(x - 5)(x - 2)(x + 1) > 0$
- $(x - 4)(x + 2)^2 < 0$
- $(x + 3)^2(x + 1) > 0$
- $(x - 2)^2(x - 3)(x + 1) \leq 0$
- $x^2(x^2 - 1) \geq 0$
- $x^3 - 4x > 0$
- $16x \leq x^3$
- $\frac{x - 3}{x + 1} \geq 0$
- $\frac{2x + 6}{x - 2} < 0$
- $\frac{4x}{2x + 3} > 2$
- $-2 < \frac{x + 1}{x - 3}$
- $\frac{2x + 1}{x - 5} \leq 3$
- $\frac{3 + x}{3 - x} \geq 1$
- $\frac{4}{x} < x$
- $\frac{x}{x + 1} > 3x$
- $1 + \frac{2}{x + 1} \leq \frac{2}{x}$
- $\frac{3}{x - 1} - \frac{4}{x} \geq 1$
- $\frac{6}{x - 1} - \frac{6}{x} \geq 1$
- $\frac{x}{2} \geq \frac{5}{x + 1} + 4$
- $\frac{x + 2}{x + 3} < \frac{x - 1}{x - 2}$
- $\frac{1}{x + 1} + \frac{1}{x + 2} \leq 0$
- $x^4 > x^2$
- $x^5 > x^2$

73–88 ■ Solve the absolute value inequality. Express the answer using interval notation and graph the solution set.

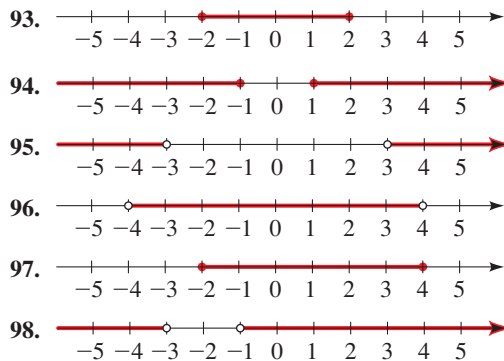
- $|x| \leq 4$
- $|3x| < 15$
- $|2x| > 7$
- $\frac{1}{2}|x| \geq 1$
- $|x - 5| \leq 3$
- $|x + 1| \geq 1$
- $|2x - 3| \leq 0.4$
- $|5x - 2| < 6$
- $|3x - 2| \geq 5$
- $|8x + 3| > 12$
- $\left| \frac{x - 2}{3} \right| < 2$
- $\left| \frac{x + 1}{2} \right| \geq 4$
- $|x + 6| < 0.001$
- $3 - |2x + 4| \leq 1$
- $8 - |2x - 1| \geq 6$
- $7|x + 2| + 5 > 4$

89–92 ■ A phrase describing a set of real numbers is given. Express the phrase as an inequality involving an absolute value.

- All real numbers x less than 3 units from 0

90. All real numbers x more than 2 units from 0
 91. All real numbers x at least 5 units from 7
 92. All real numbers x at most 4 units from 2

93–98 ■ A set of real numbers is graphed. Find an inequality involving an absolute value that describes the set.



99–102 ■ Determine the values of the variable for which the expression is defined as a real number.

99. $\sqrt{16 - 9x^2}$ 100. $\sqrt{3x^2 - 5x + 2}$
 101. $\left(\frac{1}{x^2 - 5x - 14}\right)^{1/2}$ 102. $\sqrt[4]{\frac{1-x}{2+x}}$

103. Solve the inequality for x , assuming that a , b , and c are positive constants.

(a) $a(bx - c) \geq bc$ (b) $a \leq bx + c < 2a$

104. Suppose that a , b , c , and d are positive numbers such that

$$\frac{a}{b} < \frac{c}{d}$$

Show that $\frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}$.

APPLICATIONS

-  **105. Temperature Scales** Use the relationship between C and F given in Example 9 to find the interval on the Fahrenheit scale corresponding to the temperature range $20 \leq C \leq 30$.
- 106. Temperature Scales** What interval on the Celsius scale corresponds to the temperature range $50 \leq F \leq 95$?
-  **107. Car Rental Cost** A car rental company offers two plans for renting a car.
- Plan A: \$30 per day and 10¢ per mile
 Plan B: \$50 per day with free unlimited mileage
- For what range of miles will Plan B save you money?
- 108. Long-Distance Cost** A telephone company offers two long-distance plans.
- Plan A: \$25 per month and 5¢ per minute
 Plan B: \$5 per month and 12¢ per minute
- For how many minutes of long-distance calls would Plan B be financially advantageous?

109. Driving Cost It is estimated that the annual cost of driving a certain new car is given by the formula

$$C = 0.35m + 2200$$

where m represents the number of miles driven per year and C is the cost in dollars. Jane has purchased such a car and decides to budget between \$6400 and \$7100 for next year's driving costs. What is the corresponding range of miles that she can drive her new car?

110. Air Temperature As dry air moves upward, it expands and, in so doing, cools at a rate of about 1°C for each 100-meter rise, up to about 12 km.

- (a) If the ground temperature is 20°C , write a formula for the temperature at height h .
 (b) What range of temperatures can be expected if a plane takes off and reaches a maximum height of 5 km?

111. Airline Ticket Price A charter airline finds that on its Saturday flights from Philadelphia to London all 120 seats will be sold if the ticket price is \$200. However, for each \$3 increase in ticket price, the number of seats sold decreases by one.

- (a) Find a formula for the number of seats sold if the ticket price is P dollars.
 (b) Over a certain period the number of seats sold for this flight ranged between 90 and 115. What was the corresponding range of ticket prices?

112. Accuracy of a Scale A coffee merchant sells a customer 3 lb of Hawaiian Kona at \$6.50 per pound. The merchant's scale is accurate to within ± 0.03 lb. By how much could the customer have been overcharged or undercharged because of possible inaccuracy in the scale?

113. Gravity The gravitational force F exerted by the earth on an object having a mass of 100 kg is given by the equation

$$F = \frac{4,000,000}{d^2}$$

where d is the distance (in km) of the object from the center of the earth, and the force F is measured in newtons (N). For what distances will the gravitational force exerted by the earth on this object be between 0.0004 N and 0.01 N?

114. Bonfire Temperature In the vicinity of a bonfire the temperature T in $^\circ\text{C}$ at a distance of x meters from the center of the fire was given by

$$T = \frac{600,000}{x^2 + 300}$$

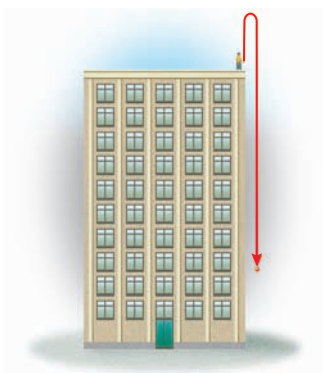
At what range of distances from the fire's center was the temperature less than 500°C ?



- 115. Falling Ball** Using calculus, it can be shown that if a ball is thrown upward with an initial velocity of 16 ft/s from the top of a building 128 ft high, then its height h above the ground t seconds later will be

$$h = 128 + 16t - 16t^2$$

During what time interval will the ball be at least 32 ft above the ground?

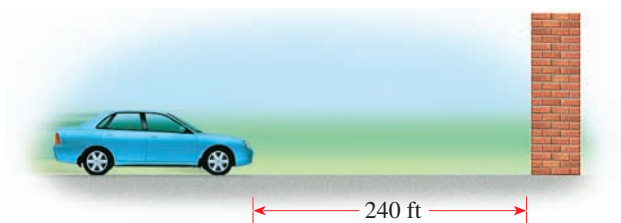


- 116. Gas Mileage** The gas mileage g (measured in mi/gal) for a particular vehicle, driven at v mi/h, is given by the formula $g = 10 + 0.9v - 0.01v^2$, as long as v is between 10 mi/h and 75 mi/h. For what range of speeds is the vehicle's mileage 30 mi/gal or better?

- 117. Stopping Distance** For a certain model of car the distance d required to stop the vehicle if it is traveling at v mi/h is given by the formula

$$d = v + \frac{v^2}{20}$$

where d is measured in feet. Kerry wants her stopping distance not to exceed 240 ft. At what range of speeds can she travel?



- 118. Manufacturer's Profit** If a manufacturer sells x units of a certain product, revenue R and cost C (in dollars) are given by

$$R = 20x$$

$$C = 2000 + 8x + 0.0025x^2$$

Use the fact that

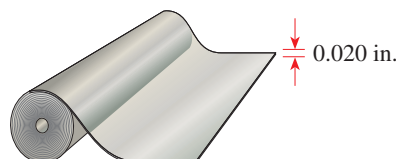
$$\text{profit} = \text{revenue} - \text{cost}$$

to determine how many units the manufacturer should sell to enjoy a profit of at least \$2400.

- 119. Fencing a Garden** A determined gardener has 120 ft of deer-resistant fence. She wants to enclose a rectangular vegetable garden in her backyard, and she wants the area that is enclosed to be at least 800 ft². What range of values is possible for the length of her garden?

- 120. Thickness of a Laminate** A company manufactures industrial laminates (thin nylon-based sheets) of thickness 0.020 in, with a tolerance of 0.003 in.

- (a) Find an inequality involving absolute values that describes the range of possible thickness for the laminate.
(b) Solve the inequality you found in part (a).



- 121. Range of Height** The average height of adult males is 68.2 in, and 95% of adult males have height h that satisfies the inequality

$$\left| \frac{h - 68.2}{2.9} \right| \leq 2$$

Solve the inequality to find the range of heights.

DISCOVERY ■ DISCUSSION ■ WRITING

- 122. Do Powers Preserve Order?** If $a < b$, is $a^2 < b^2$? (Check both positive and negative values for a and b .) If $a < b$, is $a^3 < b^3$? On the basis of your observations, state a general rule about the relationship between a^n and b^n when $a < b$ and n is a positive integer.

- 123. What's Wrong Here?** It is tempting to try to solve an inequality like an equation. For instance, we might try to solve $1 < 3/x$ by multiplying both sides by x , to get $x < 3$, so the solution would be $(-\infty, 3)$. But that's wrong; for example, $x = -1$ lies in this interval but does not satisfy the original inequality. Explain why this method doesn't work (think about the *sign* of x). Then solve the inequality correctly.

- 124. Using Distances to Solve Absolute Value Inequalities** Recall that $|a - b|$ is the distance between a and b on the number line. For any number x , what do $|x - 1|$ and $|x - 3|$ represent? Use this interpretation to solve the inequality $|x - 1| < |x - 3|$ geometrically. In general, if $a < b$, what is the solution of the inequality $|x - a| < |x - b|$?

1.8 COORDINATE GEOMETRY

The Coordinate Plane ► The Distance and Midpoint Formulas ► Graphs of Equations in Two Variables ► Intercepts ► Circles ► Symmetry

The *coordinate plane* is the link between algebra and geometry. In the coordinate plane we can draw graphs of algebraic equations. The graphs, in turn, allow us to “see” the relationship between the variables in the equation. In this section we study the coordinate plane.

▼ The Coordinate Plane

The Cartesian plane is named in honor of the French mathematician René Descartes (1596–1650), although another Frenchman, Pierre Fermat (1601–1665), also invented the principles of coordinate geometry at the same time. (See their biographies on pages 181 and 99.)

Just as points on a line can be identified with real numbers to form the coordinate line, points in a plane can be identified with ordered pairs of numbers to form the **coordinate plane** or **Cartesian plane**. To do this, we draw two perpendicular real lines that intersect at 0 on each line. Usually, one line is horizontal with positive direction to the right and is called the **x-axis**; the other line is vertical with positive direction upward and is called the **y-axis**. The point of intersection of the x-axis and the y-axis is the **origin** O , and the two axes divide the plane into four **quadrants**, labeled I, II, III, and IV in Figure 1. (The points *on* the coordinate axes are not assigned to any quadrant.)

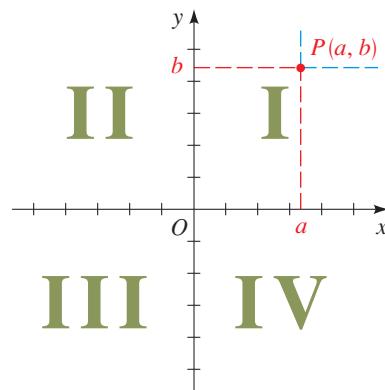


FIGURE 1

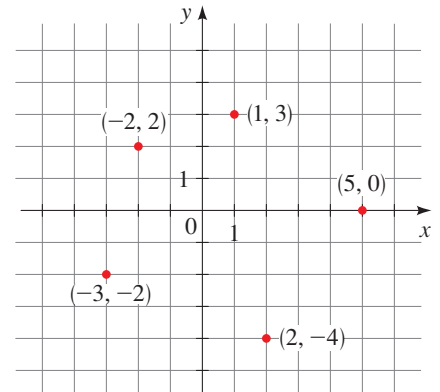


FIGURE 2

Although the notation for a point (a, b) is the same as the notation for an open interval (a, b) , the context should make clear which meaning is intended.

Any point P in the coordinate plane can be located by a unique **ordered pair** of numbers (a, b) , as shown in Figure 1. The first number a is called the **x-coordinate** of P ; the second number b is called the **y-coordinate** of P . We can think of the coordinates of P as its “address,” because they specify its location in the plane. Several points are labeled with their coordinates in Figure 2.

EXAMPLE 1 | Graphing Regions in the Coordinate Plane

Describe and sketch the regions given by each set.

- (a) $\{(x, y) \mid x \geq 0\}$ (b) $\{(x, y) \mid y = 1\}$ (c) $\{(x, y) \mid |y| < 1\}$

SOLUTION

- (a) The points whose x -coordinates are 0 or positive lie on the y -axis or to the right of it, as shown in Figure 3(a).
 (b) The set of all points with y -coordinate 1 is a horizontal line one unit above the x -axis, as in Figure 3(b).

Coordinates as Addresses

The coordinates of a point in the xy -plane uniquely determine its location. We can think of the coordinates as the “address” of the point. In Salt Lake City, Utah, the addresses of most buildings are in fact expressed as coordinates. The city is divided into quadrants with Main Street as the vertical (North-South) axis and S. Temple Street as the horizontal (East-West) axis. An address such as

1760 W 2100 S

indicates a location 17.6 blocks west of Main Street and 21 blocks south of S. Temple Street. (This is the address of the main post office in Salt Lake City.) With this logical system it is possible for someone unfamiliar with the city to locate any address immediately, as easily as one locates a point in the coordinate plane.



(c) Recall from Section 1.7 that

$$|y| < 1 \quad \text{if and only if} \quad -1 < y < 1$$

So the given region consists of those points in the plane whose y -coordinates lie between -1 and 1 . Thus, the region consists of all points that lie between (but not on) the horizontal lines $y = 1$ and $y = -1$. These lines are shown as broken lines in Figure 3(c) to indicate that the points on these lines do not lie in the set.

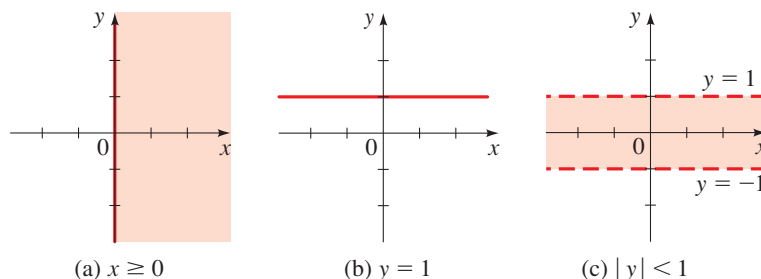


FIGURE 3

NOW TRY EXERCISES 23, 25, AND 29

▼ The Distance and Midpoint Formulas

We now find a formula for the distance $d(A, B)$ between two points $A(x_1, y_1)$ and $B(x_2, y_2)$ in the plane. Recall from Section 1.1 that the distance between points a and b on a number line is $d(a, b) = |b - a|$. So from Figure 4 we see that the distance between the points $A(x_1, y_1)$ and $C(x_2, y_1)$ on a horizontal line must be $|x_2 - x_1|$, and the distance between $B(x_2, y_2)$ and $C(x_2, y_1)$ on a vertical line must be $|y_2 - y_1|$.

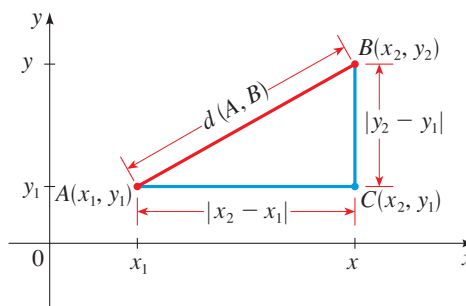


FIGURE 4

Since triangle ABC is a right triangle, the Pythagorean Theorem gives

$$d(A, B) = \sqrt{|x_2 - x_1|^2 + |y_2 - y_1|^2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

DISTANCE FORMULA

The distance between the points $A(x_1, y_1)$ and $B(x_2, y_2)$ in the plane is

$$d(A, B) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

EXAMPLE 2 | Applying the Distance Formula

Which of the points $P(1, -2)$ or $Q(8, 9)$ is closer to the point $A(5, 3)$?

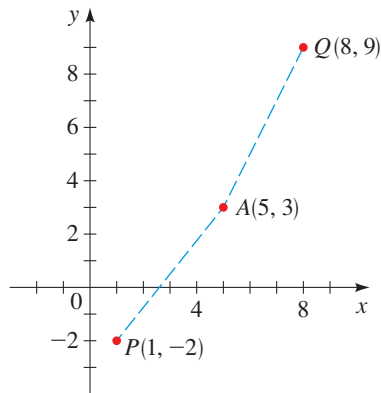


FIGURE 5

SOLUTION By the Distance Formula we have

$$d(P, A) = \sqrt{(5 - 1)^2 + [3 - (-2)]^2} = \sqrt{4^2 + 5^2} = \sqrt{41}$$

$$d(Q, A) = \sqrt{(5 - 8)^2 + (3 - 9)^2} = \sqrt{(-3)^2 + (-6)^2} = \sqrt{45}$$

This shows that $d(P, A) < d(Q, A)$, so P is closer to A (see Figure 5).

NOW TRY EXERCISE 33

Now let's find the coordinates (x, y) of the midpoint M of the line segment that joins the point $A(x_1, y_1)$ to the point $B(x_2, y_2)$. In Figure 6 notice that triangles APM and MQB are congruent because $d(A, M) = d(M, B)$ and the corresponding angles are equal.

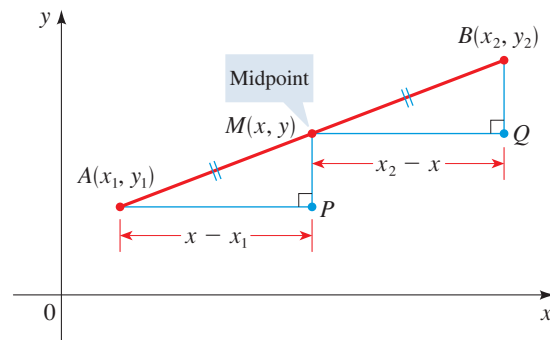


FIGURE 6

It follows that $d(A, P) = d(M, Q)$, so

$$x - x_1 = x_2 - x$$

Solving this equation for x , we get $2x = x_1 + x_2$, so $x = \frac{x_1 + x_2}{2}$. Similarly, $y = \frac{y_1 + y_2}{2}$.

MIDPOINT FORMULA

The midpoint of the line segment from $A(x_1, y_1)$ to $B(x_2, y_2)$ is

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right)$$

EXAMPLE 3 | Applying the Midpoint Formula

Show that the quadrilateral with vertices $P(1, 2)$, $Q(4, 4)$, $R(5, 9)$, and $S(2, 7)$ is a parallelogram by proving that its two diagonals bisect each other.

SOLUTION If the two diagonals have the same midpoint, then they must bisect each other. The midpoint of the diagonal PR is

$$\left(\frac{1 + 5}{2}, \frac{2 + 9}{2} \right) = \left(3, \frac{11}{2} \right)$$

and the midpoint of the diagonal QS is

$$\left(\frac{4 + 2}{2}, \frac{4 + 7}{2} \right) = \left(3, \frac{11}{2} \right)$$

so each diagonal bisects the other, as shown in Figure 7. (A theorem from elementary geometry states that the quadrilateral is therefore a parallelogram.)

NOW TRY EXERCISE 37

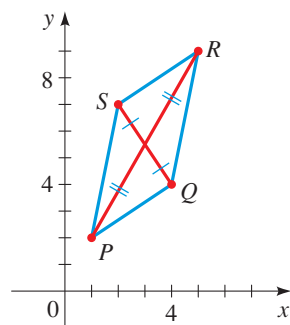


FIGURE 7

Fundamental Principle of Analytic Geometry

A point (x, y) lies on the graph of an equation if and only if its coordinates satisfy the equation.

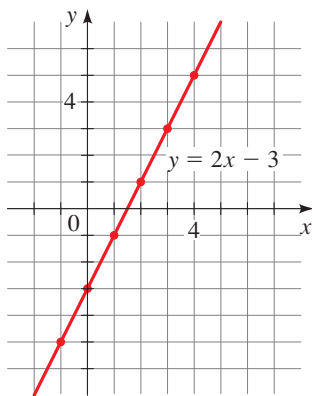


FIGURE 8

A detailed discussion of parabolas and their geometric properties is presented in Chapter 10.

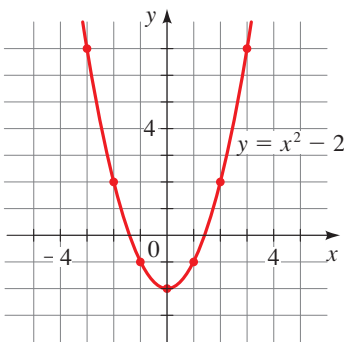


FIGURE 9

Graphs of Equations in Two Variables

An **equation in two variables**, such as $y = x^2 + 1$, expresses a relationship between two quantities. A point (x, y) **satisfies** the equation if it makes the equation true when the values for x and y are substituted into the equation. For example, the point $(3, 10)$ satisfies the equation $y = x^2 + 1$ because $10 = 3^2 + 1$, but the point $(1, 3)$ does not, because $3 \neq 1^2 + 1$.

THE GRAPH OF AN EQUATION

The **graph** of an equation in x and y is the set of all points (x, y) in the coordinate plane that satisfy the equation.

The graph of an equation is a curve, so to graph an equation, we plot as many points as we can, then connect them by a smooth curve.

EXAMPLE 4 | Sketching a Graph by Plotting Points

Sketch the graph of the equation $2x - y = 3$.

SOLUTION We first solve the given equation for y to get

$$y = 2x - 3$$

This helps us calculate the y -coordinates in the following table.

x	$y = 2x - 3$	(x, y)
-1	-5	$(-1, -5)$
0	-3	$(0, -3)$
1	-1	$(1, -1)$
2	1	$(2, 1)$
3	3	$(3, 3)$
4	5	$(4, 5)$

Of course, there are infinitely many points on the graph, and it is impossible to plot all of them. But the more points we plot, the better we can imagine what the graph represented by the equation looks like. We plot the points we found in Figure 8; they appear to lie on a line. So we complete the graph by joining the points by a line. (In Section 1.10 we verify that the graph of this equation is indeed a line.)

 **NOW TRY EXERCISE 59**

EXAMPLE 5 | Sketching a Graph by Plotting Points

Sketch the graph of the equation $y = x^2 - 2$.

SOLUTION We find some of the points that satisfy the equation in the following table. In Figure 9 we plot these points and then connect them by a smooth curve. A curve with this shape is called a *parabola*.

x	$y = x^2 - 2$	(x, y)
-3	7	$(-3, 7)$
-2	2	$(-2, 2)$
-1	-1	$(-1, -1)$
0	-2	$(0, -2)$
1	-1	$(1, -1)$
2	2	$(2, 2)$
3	7	$(3, 7)$

 **NOW TRY EXERCISE 63**

EXAMPLE 6 | Graphing an Absolute Value Equation

Sketch the graph of the equation $y = |x|$.

SOLUTION We make a table of values:

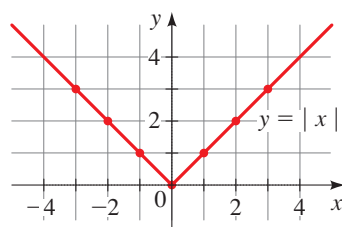


FIGURE 10

x	$y = x $	(x, y)
-3	3	$(-3, 3)$
-2	2	$(-2, 2)$
-1	1	$(-1, 1)$
0	0	$(0, 0)$
1	1	$(1, 1)$
2	2	$(2, 2)$
3	3	$(3, 3)$

In Figure 10 we plot these points and use them to sketch the graph of the equation.

NOW TRY EXERCISE 75

Intercepts

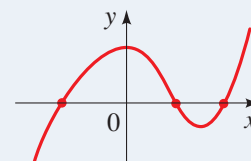
The x -coordinates of the points where a graph intersects the x -axis are called the **x -intercepts** of the graph and are obtained by setting $y = 0$ in the equation of the graph. The y -coordinates of the points where a graph intersects the y -axis are called the **y -intercepts** of the graph and are obtained by setting $x = 0$ in the equation of the graph.

DEFINITION OF INTERCEPTS**Intercepts** **x -intercepts:**

The x -coordinates of points where the graph of an equation intersects the x -axis

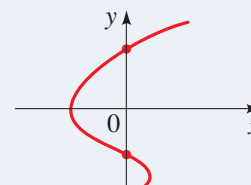
How to find them

Set $y = 0$ and solve for x

Where they are on the graph **y -intercepts:**

The y -coordinates of points where the graph of an equation intersects the y -axis

Set $x = 0$ and solve for y

**EXAMPLE 7** | Finding Intercepts

Find the x - and y -intercepts of the graph of the equation $y = x^2 - 2$.

SOLUTION To find the x -intercepts, we set $y = 0$ and solve for x . Thus

$$0 = x^2 - 2 \quad \text{Set } y = 0$$

$$x^2 = 2 \quad \text{Add 2 to each side}$$

$$x = \pm\sqrt{2} \quad \text{Take the square root}$$

The x -intercepts are $\sqrt{2}$ and $-\sqrt{2}$.

To find the y -intercepts, we set $x = 0$ and solve for y . Thus

$$y = 0^2 - 2 \quad \text{Set } x = 0$$

$$y = -2$$

The y -intercept is -2 .

The graph of this equation was sketched in Example 5. It is repeated in Figure 11 with the x - and y -intercepts labeled.

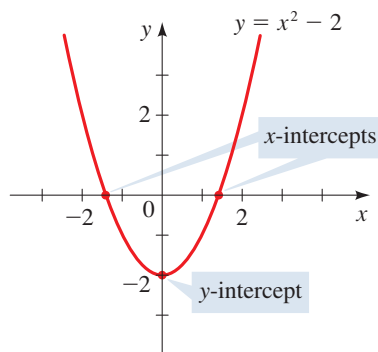


FIGURE 11

NOW TRY EXERCISE 65

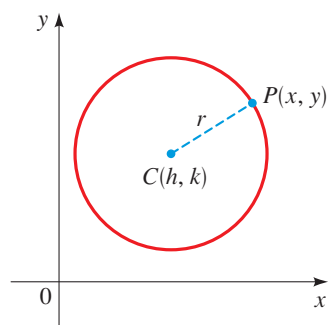


FIGURE 12

▼ Circles

So far, we have discussed how to find the graph of an equation in x and y . The converse problem is to find an equation of a graph, that is, an equation that represents a given curve in the xy -plane. Such an equation is satisfied by the coordinates of the points on the curve and by no other point. This is the other half of the fundamental principle of analytic geometry as formulated by Descartes and Fermat. The idea is that if a geometric curve can be represented by an algebraic equation, then the rules of algebra can be used to analyze the curve.

As an example of this type of problem, let's find the equation of a circle with radius r and center (h, k) . By definition the circle is the set of all points $P(x, y)$ whose distance from the center $C(h, k)$ is r (see Figure 12). Thus P is on the circle if and only if $d(P, C) = r$. From the distance formula we have

$$\sqrt{(x - h)^2 + (y - k)^2} = r$$

$$(x - h)^2 + (y - k)^2 = r^2 \quad \text{Square each side}$$

This is the desired equation.

EQUATION OF A CIRCLE

An equation of the circle with center (h, k) and radius r is

$$(x - h)^2 + (y - k)^2 = r^2$$

This is called the **standard form** for the equation of the circle. If the center of the circle is the origin $(0, 0)$, then the equation is

$$x^2 + y^2 = r^2$$

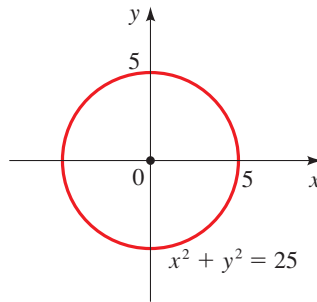
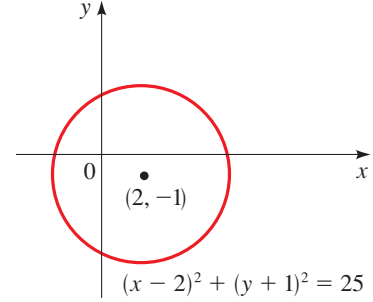
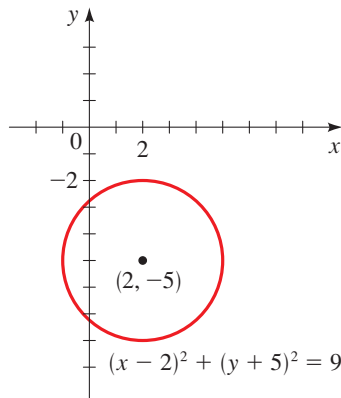
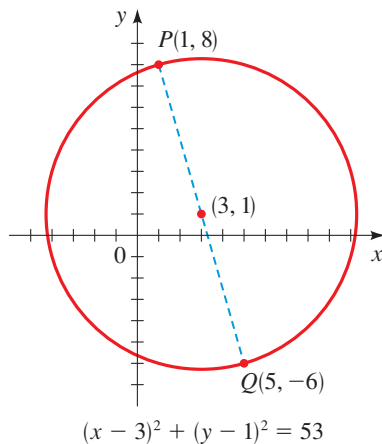
EXAMPLE 8 | Graphing a Circle

Graph each equation.

(a) $x^2 + y^2 = 25$ (b) $(x - 2)^2 + (y + 1)^2 = 25$

SOLUTION

- (a) Rewriting the equation as $x^2 + y^2 = 5^2$, we see that this is an equation of the circle of radius 5 centered at the origin. Its graph is shown in Figure 13.
- (b) Rewriting the equation as $(x - 2)^2 + (y + 1)^2 = 5^2$, we see that this is an equation of the circle of radius 5 centered at $(2, -1)$. Its graph is shown in Figure 14.

**FIGURE 13****FIGURE 14****FIGURE 15****FIGURE 16**

NOW TRY EXERCISES 87 AND 89

EXAMPLE 9 | Finding an Equation of a Circle

- (a) Find an equation of the circle with radius 3 and center $(2, -5)$.
- (b) Find an equation of the circle that has the points $P(1, 8)$ and $Q(5, -6)$ as the endpoints of a diameter.

SOLUTION

- (a) Using the equation of a circle with $r = 3$, $h = 2$, and $k = -5$, we obtain

$$(x - 2)^2 + (y + 5)^2 = 9$$

The graph is shown in Figure 15.

- (b) We first observe that the center is the midpoint of the diameter PQ , so by the Midpoint Formula the center is

$$\left(\frac{1 + 5}{2}, \frac{8 - 6}{2} \right) = (3, 1)$$

The radius r is the distance from P to the center, so by the Distance Formula

$$r^2 = (3 - 1)^2 + (1 - 8)^2 = 2^2 + (-7)^2 = 53$$

Therefore, the equation of the circle is

$$(x - 3)^2 + (y - 1)^2 = 53$$

The graph is shown in Figure 16.

NOW TRY EXERCISES 93 AND 97

Let's expand the equation of the circle in the preceding example.

$$(x - 3)^2 + (y - 1)^2 = 53 \quad \text{Standard form}$$

$$x^2 - 6x + 9 + y^2 - 2y + 1 = 53 \quad \text{Expand the squares}$$

$$x^2 - 6x + y^2 - 2y = 43 \quad \text{Subtract 10 to get expanded form}$$

Suppose we are given the equation of a circle in expanded form. Then to find its center and radius, we must put the equation back in standard form. That means that we must reverse the steps in the preceding calculation, and to do that we need to know what to add to an expression like $x^2 - 6x$ to make it a perfect square—that is, we need to complete the square, as in the next example.

Completing the square is used in many contexts in algebra. In Section 1.5 we used completing the square to solve quadratic equations.

EXAMPLE 10 | Identifying an Equation of a Circle

Show that the equation $x^2 + y^2 + 2x - 6y + 7 = 0$ represents a circle, and find the center and radius of the circle.

SOLUTION We first group the x -terms and y -terms. Then we complete the square within each grouping. That is, we complete the square for $x^2 + 2x$ by adding $(\frac{1}{2} \cdot 2)^2 = 1$, and we complete the square for $y^2 - 6y$ by adding $[\frac{1}{2} \cdot (-6)]^2 = 9$.

$$(x^2 + 2x \quad) + (y^2 - 6y \quad) = -7$$

Group terms

$$(x^2 + 2x + 1) + (y^2 - 6y + 9) = -7 + 1 + 9$$

Complete the square by adding 1 and 9 to each side

$$(x + 1)^2 + (y - 3)^2 = 3$$

Factor and simplify

Comparing this equation with the standard equation of a circle, we see that $h = -1$, $k = 3$, and $r = \sqrt{3}$, so the given equation represents a circle with center $(-1, 3)$ and radius $\sqrt{3}$.

 **NOW TRY EXERCISE 103**

 We must add the same numbers to each side to maintain equality.

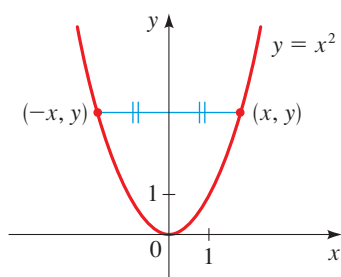


FIGURE 17

Symmetry

Figure 17 shows the graph of $y = x^2$. Notice that the part of the graph to the left of the y -axis is the mirror image of the part to the right of the y -axis. The reason is that if the point (x, y) is on the graph, then so is $(-x, y)$, and these points are reflections of each other about the y -axis. In this situation we say that the graph is **symmetric with respect to the y -axis**. Similarly, we say that a graph is **symmetric with respect to the x -axis** if whenever the point (x, y) is on the graph, then so is $(x, -y)$. A graph is **symmetric with respect to the origin** if whenever (x, y) is on the graph, so is $(-x, -y)$.

DEFINITION OF SYMMETRY

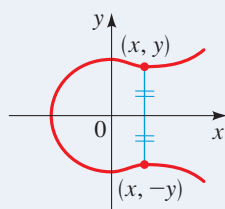
Type of symmetry

Symmetry with respect to the x -axis

How to test for symmetry

The equation is unchanged when y is replaced by $-y$

What the graph looks like (figures in this section)



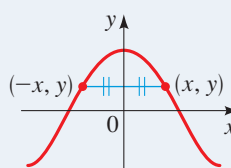
(Figures 13, 18)

Geometric meaning

Graph is unchanged when reflected in the x -axis

Symmetry with respect to the y -axis

The equation is unchanged when x is replaced by $-x$

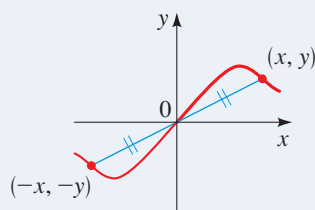


(Figures 9, 10, 11, 13, 17)

Graph is unchanged when reflected in the y -axis

Symmetry with respect to the origin

The equation is unchanged when x is replaced by $-x$ and y by $-y$



(Figures 13, 19)

Graph is unchanged when rotated 180° about the origin

The remaining examples in this section show how symmetry helps us sketch the graphs of equations.

EXAMPLE 11 | Using Symmetry to Sketch a Graph

Test the equation $x = y^2$ for symmetry and sketch the graph.

SOLUTION If y is replaced by $-y$ in the equation $x = y^2$, we get

$$x = (-y)^2 \quad \text{Replace } y \text{ by } -y$$

$$x = y^2 \quad \text{Simplify}$$

and so the equation is unchanged. Therefore, the graph is symmetric about the x -axis. But changing x to $-x$ gives the equation $-x = y^2$, which is not the same as the original equation, so the graph is not symmetric about the y -axis.

We use the symmetry about the x -axis to sketch the graph by first plotting points just for $y > 0$ and then reflecting the graph in the x -axis, as shown in Figure 18.

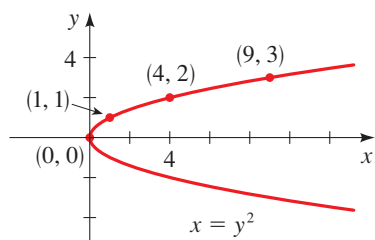


FIGURE 18

y	$x = y^2$	(x, y)
0	0	(0, 0)
1	1	(1, 1)
2	4	(4, 2)
3	9	(9, 3)

NOW TRY EXERCISE 77

EXAMPLE 12 | Using Symmetry to Sketch a Graph

Test the equation $y = x^3 - 9x$ for symmetry and sketch its graph.

SOLUTION If we replace x by $-x$ and y by $-y$ in the equation, we get

$$-y = (-x)^3 - 9(-x) \quad \text{Replace } x \text{ by } -x \text{ and } y \text{ by } -y$$

$$-y = -x^3 + 9x \quad \text{Simplify}$$

$$y = x^3 - 9x \quad \text{Multiply by } -1$$

and so the equation is unchanged. This means that the graph is symmetric with respect to the origin. We sketch it by first plotting points for $x > 0$ and then using symmetry about the origin (see Figure 19).

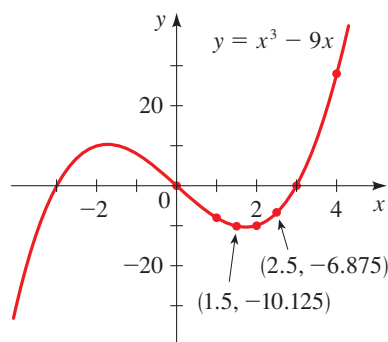


FIGURE 19

x	$y = x^3 - 9x$	(x, y)
0	0	(0, 0)
1	-8	(1, -8)
1.5	-10.125	(1.5, -10.125)
2	-10	(2, -10)
2.5	-6.875	(2.5, -6.875)
3	0	(3, 0)
4	28	(4, 28)

NOW TRY EXERCISE 79

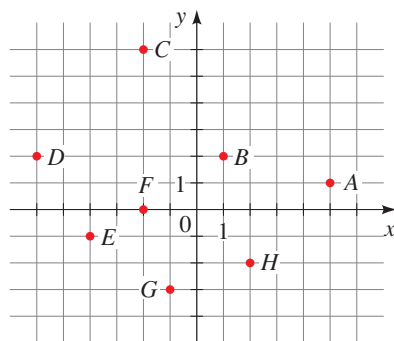
1.8 EXERCISES

CONCEPTS

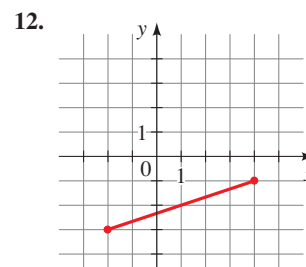
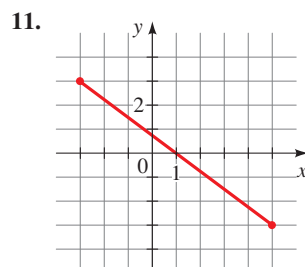
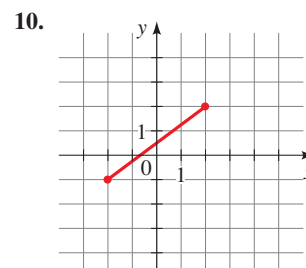
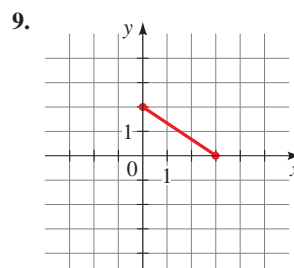
- The point that is 3 units to the right of the y -axis and 5 units below the x -axis has coordinates $(\rule{1cm}{0.4pt}, \rule{1cm}{0.4pt})$.
- The distance between the points (a, b) and (c, d) is $\rule{1cm}{0.4pt}$. So the distance between $(1, 2)$ and $(7, 10)$ is $\rule{1cm}{0.4pt}$.
- The point midway between (a, b) and (c, d) is $\rule{1cm}{0.4pt}$. So the point midway between $(1, 2)$ and $(7, 10)$ is $\rule{1cm}{0.4pt}$.
- If the point $(2, 3)$ is on the graph of an equation in x and y , then the equation is satisfied when we replace x by $\rule{1cm}{0.4pt}$ and y by $\rule{1cm}{0.4pt}$. Is the point $(2, 3)$ on the graph of the equation $2y = x + 1$?
- (a) To find the x -intercept(s) of the graph of an equation, we set $\rule{1cm}{0.4pt}$ equal to 0 and solve for $\rule{1cm}{0.4pt}$. So the x -intercept of $2y = x + 1$ is $\rule{1cm}{0.4pt}$.
(b) To find the y -intercept(s) of the graph of an equation, we set $\rule{1cm}{0.4pt}$ equal to 0 and solve for $\rule{1cm}{0.4pt}$. So the y -intercept of $2y = x + 1$ is $\rule{1cm}{0.4pt}$.
- The graph of the equation $(x - 1)^2 + (y - 2)^2 = 9$ is a circle with center $(\rule{1cm}{0.4pt}, \rule{1cm}{0.4pt})$ and radius $\rule{1cm}{0.4pt}$.

SKILLS

- Plot the given points in a coordinate plane.
 $(2, 3), (-2, 3), (4, 5), (4, -5), (-4, 5), (-4, -5)$
- Find the coordinates of the points shown in the figure.



- 9–12 ■** A pair of points is graphed.
- Find the distance between them.
 - Find the midpoint of the segment that joins them.



13–18 ■ A pair of points is graphed.

- Plot the points in a coordinate plane.
- Find the distance between them.
- Find the midpoint of the segment that joins them.

13. $(0, 8), (6, 16)$ **14.** $(-2, 5), (10, 0)$

15. $(-3, -6), (4, 18)$ **16.** $(-1, -1), (9, 9)$

17. $(6, -2), (-6, 2)$ **18.** $(0, -6), (5, 0)$

19. Draw the rectangle with vertices $A(1, 3)$, $B(5, 3)$, $C(1, -3)$, and $D(5, -3)$ on a coordinate plane. Find the area of the rectangle.

20. Draw the parallelogram with vertices $A(1, 2)$, $B(5, 2)$, $C(3, 6)$, and $D(7, 6)$ on a coordinate plane. Find the area of the parallelogram.

21. Plot the points $A(1, 0)$, $B(5, 0)$, $C(4, 3)$, and $D(2, 3)$ on a coordinate plane. Draw the segments AB , BC , CD , and DA . What kind of quadrilateral is $ABCD$, and what is its area?

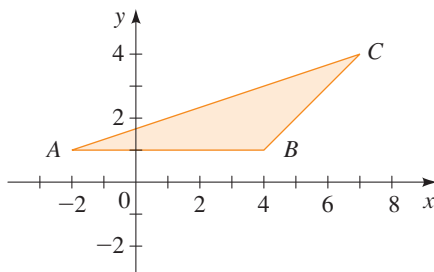
22. Plot the points $P(5, 1)$, $Q(0, 6)$, and $R(-5, 1)$ on a coordinate plane. Where must the point S be located so that the quadrilateral $PQRS$ is a square? Find the area of this square.

23–32 ■ Sketch the region given by the set.

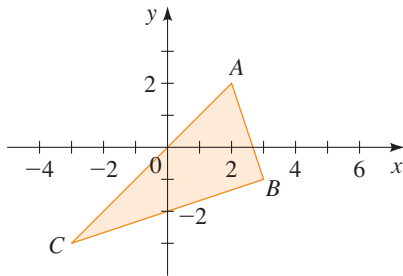
- $\{(x, y) \mid x \geq 3\}$
- $\{(x, y) \mid y < 3\}$
- $\{(x, y) \mid y = 2\}$
- $\{(x, y) \mid x = -1\}$
- $\{(x, y) \mid 1 < x < 2\}$
- $\{(x, y) \mid 0 \leq y \leq 4\}$
- $\{(x, y) \mid |x| > 4\}$
- $\{(x, y) \mid |y| \leq 2\}$
- $\{(x, y) \mid x \geq 1 \text{ and } y < 3\}$
- $\{(x, y) \mid |x| \leq 2 \text{ and } |y| \leq 3\}$

33. Which of the points $A(6, 7)$ or $B(-5, 8)$ is closer to the origin?
34. Which of the points $C(-6, 3)$ or $D(3, 0)$ is closer to the point $E(-2, 1)$?
35. Which of the points $P(3, 1)$ or $Q(-1, 3)$ is closer to the point $R(-1, -1)$?
36. (a) Show that the points $(7, 3)$ and $(3, 7)$ are the same distance from the origin.
(b) Show that the points (a, b) and (b, a) are the same distance from the origin.
37. Show that the triangle with vertices $A(0, 2)$, $B(-3, -1)$, and $C(-4, 3)$ is isosceles.

38. Find the area of the triangle shown in the figure.

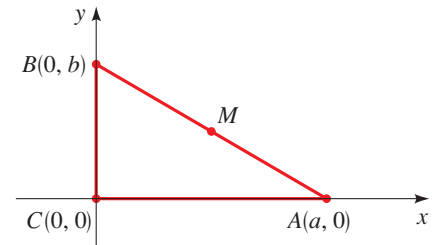


39. Refer to triangle ABC in the figure below.
(a) Show that triangle ABC is a right triangle by using the converse of the Pythagorean Theorem (see page 219).
(b) Find the area of triangle ABC .



40. Show that the triangle with vertices $A(6, -7)$, $B(11, -3)$, and $C(2, -2)$ is a right triangle by using the converse of the Pythagorean Theorem. Find the area of the triangle.
41. Show that the points $A(-2, 9)$, $B(4, 6)$, $C(1, 0)$, and $D(-5, 3)$ are the vertices of a square.
42. Show that the points $A(-1, 3)$, $B(3, 11)$, and $C(5, 15)$ are collinear by showing that $d(A, B) + d(B, C) = d(A, C)$.
43. Find a point on the y -axis that is equidistant from the points $(5, -5)$ and $(1, 1)$.
44. Find the lengths of the medians of the triangle with vertices $A(1, 0)$, $B(3, 6)$, and $C(8, 2)$. (A *median* is a line segment from a vertex to the midpoint of the opposite side.)

45. Plot the points $P(-1, -4)$, $Q(1, 1)$, and $R(4, 2)$ on a coordinate plane. Where should the point S be located so that the figure $PQRS$ is a parallelogram?
46. If $M(6, 8)$ is the midpoint of the line segment AB and if A has coordinates $(2, 3)$, find the coordinates of B .
47. (a) Sketch the parallelogram with vertices $A(-2, -1)$, $B(4, 2)$, $C(7, 7)$, and $D(1, 4)$.
(b) Find the midpoints of the diagonals of this parallelogram.
(c) From part (b) show that the diagonals bisect each other.
48. The point M in the figure is the midpoint of the line segment AB . Show that M is equidistant from the vertices of triangle ABC .



49–52 ■ Determine whether the given points are on the graph of the equation.

49. $x - 2y - 1 = 0$; $(0, 0)$, $(1, 0)$, $(-1, -1)$

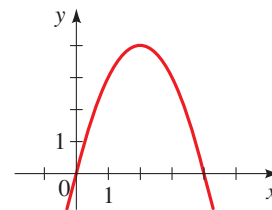
50. $y(x^2 + 1) = 1$; $(1, 1)$, $(1, \frac{1}{2})$, $(-1, \frac{1}{2})$

51. $x^2 + xy + y^2 = 4$; $(0, -2)$, $(1, -2)$, $(2, -2)$

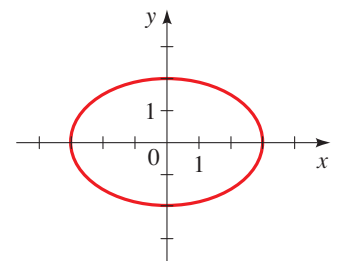
52. $x^2 + y^2 = 1$; $(0, 1)$, $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$, $(\frac{\sqrt{3}}{2}, \frac{1}{2})$

53–56 ■ An equation and its graph are given. Find the x - and y -intercepts.

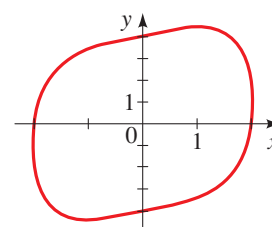
53. $y = 4x - x^2$



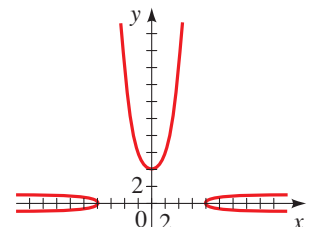
54. $\frac{x^2}{9} + \frac{y^2}{4} = 1$



55. $x^4 + y^2 - xy = 16$



56. $x^2 + y^3 - x^2y^2 = 64$



57–76 ■ Make a table of values and sketch the graph of the equation. Find the x - and y -intercepts and test for symmetry.

57. $y = -x + 4$

58. $y = 3x + 3$

59. $2x - y = 6$

60. $x + y = 3$

61. $y = 1 - x^2$

62. $y = x^2 + 2$

63. $4y = x^2$

64. $8y = x^3$

65. $y = x^2 - 9$

66. $y = 9 - x^2$

67. $xy = 2$

68. $y = \sqrt{x + 4}$

69. $y = \sqrt{4 - x^2}$

70. $y = -\sqrt{4 - x^2}$

71. $x + y^2 = 4$

72. $x = y^3$

73. $y = 16 - x^4$

74. $x = |y|$

75. $y = 4 - |x|$

76. $y = |4 - x|$

77–82 ■ Test the equation for symmetry.

77. $y = x^4 + x^2$

78. $x = y^4 - y^2$

79. $x^2y^2 + xy = 1$

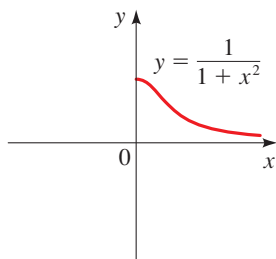
80. $x^4y^4 + x^2y^2 = 1$

81. $y = x^3 + 10x$

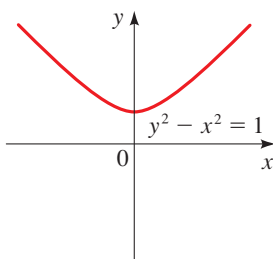
82. $y = x^2 + |x|$

83–86 ■ Complete the graph using the given symmetry property.

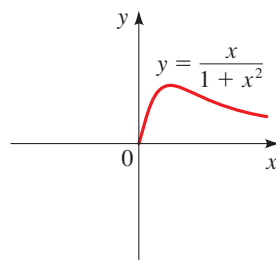
83. Symmetric with respect to the y -axis



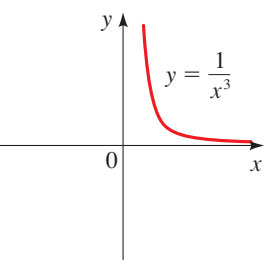
84. Symmetric with respect to the x -axis



85. Symmetric with respect to the origin



86. Symmetric with respect to the origin



87–92 ■ Find the center and radius of the circle and sketch its graph

87. $x^2 + y^2 = 9$

88. $x^2 + y^2 = 5$

89. $(x - 3)^2 + y^2 = 16$

90. $x^2 + (y - 2)^2 = 4$

91. $(x + 3)^2 + (y - 4)^2 = 25$

92. $(x + 1)^2 + (y + 2)^2 = 36$

93–100 ■ Find an equation of the circle that satisfies the given conditions.

93. Center $(2, -1)$; radius 3

94. Center $(-1, -4)$; radius 8

95. Center at the origin; passes through $(4, 7)$

96. Center $(-1, 5)$; passes through $(-4, -6)$

97. Endpoints of a diameter are $P(-1, 1)$ and $Q(5, 9)$

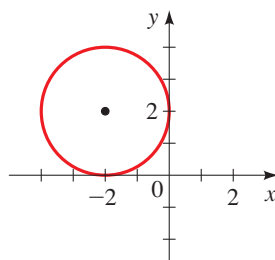
98. Endpoints of a diameter are $P(-1, 3)$ and $Q(7, -5)$

99. Center $(7, -3)$; tangent to the x -axis

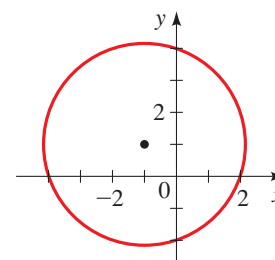
100. Circle lies in the first quadrant, tangent to both x - and y -axes; radius 5

101–102 ■ Find the equation of the circle shown in the figure.

101.



102.



103–108 ■ Show that the equation represents a circle, and find the center and radius of the circle.

103. $x^2 + y^2 - 4x + 10y + 13 = 0$

104. $x^2 + y^2 + 6y + 2 = 0$

105. $x^2 + y^2 - \frac{1}{2}x + \frac{1}{2}y = \frac{1}{8}$

106. $x^2 + y^2 + \frac{1}{2}x + 2y + \frac{1}{16} = 0$

107. $2x^2 + 2y^2 - 3x = 0$

108. $3x^2 + 3y^2 + 6x - y = 0$

109–110 ■ Sketch the region given by the set.

109. $\{(x, y) \mid x^2 + y^2 \leq 1\}$

110. $\{(x, y) \mid x^2 + y^2 > 4\}$

111. Find the area of the region that lies outside the circle $x^2 + y^2 = 4$ but inside the circle

$$x^2 + y^2 - 4y - 12 = 0$$

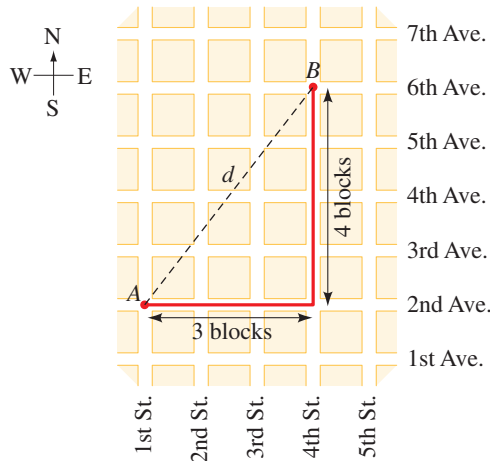
112. Sketch the region in the coordinate plane that satisfies both the inequalities $x^2 + y^2 \leq 9$ and $y \geq |x|$. What is the area of this region?

APPLICATIONS

113. Distances in a City A city has streets that run north and south and avenues that run east and west, all equally spaced. Streets and avenues are numbered sequentially, as shown in the figure. The walking distance between points A and B is 7 blocks—that is, 3 blocks east and 4 blocks north. To find the *straight-line* distance d , we must use the Distance Formula.

(a) Find the straight-line distance (in blocks) between A and B .

- (b) Find the walking distance and the straight-line distance between the corner of 4th St. and 2nd Ave. and the corner of 11th St. and 26th Ave.
- (c) What must be true about the points P and Q if the walking distance between P and Q equals the straight-line distance between P and Q ?



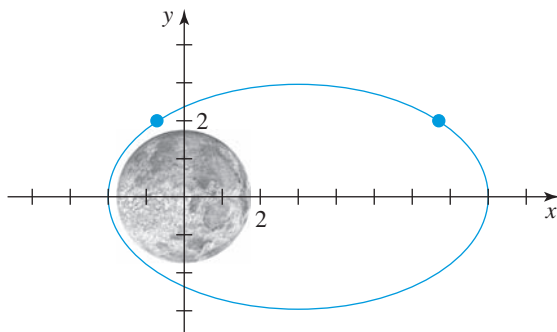
- 114. Halfway Point** Two friends live in the city described in Exercise 113, one at the corner of 3rd St. and 7th Ave., the other at the corner of 27th St. and 17th Ave. They frequently meet at a coffee shop halfway between their homes.

- (a) At what intersection is the coffee shop located?
- (b) How far must each of them walk to get to the coffee shop?

- 115. Orbit of a Satellite** A satellite is in orbit around the moon. A coordinate plane containing the orbit is set up with the center of the moon at the origin, as shown in the graph, with distances measured in megameters (Mm). The equation of the satellite's orbit is

$$\frac{(x-3)^2}{25} + \frac{y^2}{16} = 1$$

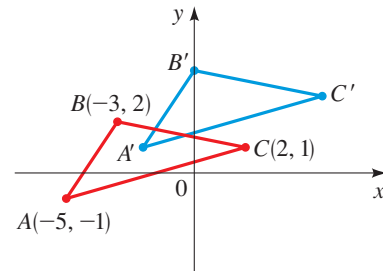
- (a) From the graph, determine the closest and the farthest that the satellite gets to the center of the moon.
- (b) There are two points in the orbit with y -coordinates 2. Find the x -coordinates of these points, and determine their distances to the center of the moon.



DISCOVERY ■ DISCUSSION ■ WRITING

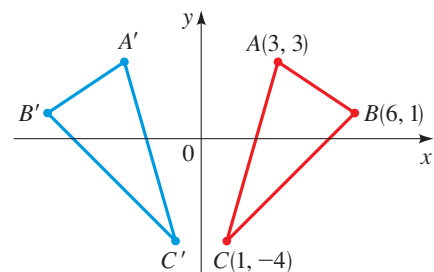
- 116. Shifting the Coordinate Plane** Suppose that each point in the coordinate plane is shifted 3 units to the right and 2 units upward.

- (a) The point $(5, 3)$ is shifted to what new point?
- (b) The point (a, b) is shifted to what new point?
- (c) What point is shifted to $(3, 4)$?
- (d) Triangle ABC in the figure has been shifted to triangle $A'B'C'$. Find the coordinates of the points A' , B' , and C' .



- 117. Reflecting in the Coordinate Plane** Suppose that the y -axis acts as a mirror that reflects each point to the right of it into a point to the left of it.

- (a) The point $(3, 7)$ is reflected to what point?
- (b) The point (a, b) is reflected to what point?
- (c) What point is reflected to $(-4, -1)$?
- (d) Triangle ABC in the figure is reflected to triangle $A'B'C'$. Find the coordinates of the points A' , B' , and C' .



- 118. Completing a Line Segment** Plot the points $M(6, 8)$ and $A(2, 3)$ on a coordinate plane. If M is the midpoint of the line segment AB , find the coordinates of B . Write a brief description of the steps you took to find B , and your reasons for taking them.

- 119. Completing a Parallelogram** Plot the points $P(0, 3)$, $Q(2, 2)$, and $R(5, 3)$ on a coordinate plane. Where should the point S be located so that the figure $PQRS$ is a parallelogram? Write a brief description of the steps you took and your reasons for taking them.

- 120. Circle, Point, or Empty Set?** Complete the squares in the general equation $x^2 + ax + y^2 + by + c = 0$ and simplify the result as much as possible. Under what conditions on the coefficients a , b , and c does this equation represent a circle? A single point? The empty set? In the case that the equation does represent a circle, find its center and radius.

121. Do the Circles Intersect?

- (a) Find the radius of each circle in the pair and the distance between their centers; then use this information to determine whether the circles intersect.

(i) $(x - 2)^2 + (y - 1)^2 = 9;$
 $(x - 6)^2 + (y - 4)^2 = 16$

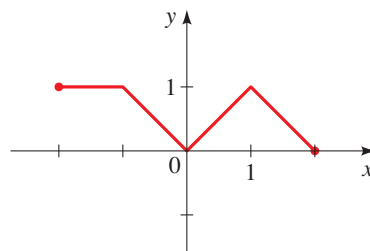
(ii) $x^2 + (y - 2)^2 = 4;$
 $(x - 5)^2 + (y - 14)^2 = 9$

(iii) $(x - 3)^2 + (y + 1)^2 = 1;$
 $(x - 2)^2 + (y - 2)^2 = 25$

- (b) How can you tell, just by knowing the radii of two circles and the distance between their centers, whether the circles intersect? Write a short paragraph describing how you would decide this and draw graphs to illustrate your answer.

- 122. Making a Graph Symmetric** The graph shown in the figure is not symmetric about the x -axis, the y -axis, or the origin. Add more line segments to the graph so that it exhibits the indicated symmetry. In each case, add as little as possible.

- (a) Symmetry about the x -axis
 (b) Symmetry about the y -axis
 (c) Symmetry about the origin



1.9 GRAPHING CALCULATORS; SOLVING EQUATIONS AND INEQUALITIES GRAPHICALLY



Using a Graphing Calculator ► Solving Equations Graphically ► Solving Inequalities Graphically

In Sections 1.5 and 1.7 we solved equations and inequalities algebraically. In Section 1.8 we learned how to sketch the graph of an equation in a coordinate plane. In this section we use graphs to solve equations and inequalities. To do this, we must first draw a graph using a graphing device. So we begin by giving a few guidelines to help us use graphing devices effectively.

▼ Using a Graphing Calculator

A graphing calculator or computer displays a rectangular portion of the graph of an equation in a display window or viewing screen, which we call a **viewing rectangle**. The default screen often gives an incomplete or misleading picture, so it is important to choose the viewing rectangle with care. If we choose the x -values to range from a minimum value of $X_{\min} = a$ to a maximum value of $X_{\max} = b$ and the y -values to range from a minimum value of $Y_{\min} = c$ to a maximum value of $Y_{\max} = d$, then the displayed portion of the graph lies in the rectangle

$$[a, b] \times [c, d] = \{(x, y) \mid a \leq x \leq b, c \leq y \leq d\}$$

as shown in Figure 1. We refer to this as the $[a, b]$ by $[c, d]$ viewing rectangle.

The graphing device draws the graph of an equation much as you would. It plots points of the form (x, y) for a certain number of values of x , equally spaced between a and b . If the equation is not defined for an x -value or if the corresponding y -value lies outside the viewing rectangle, the device ignores this value and moves on to the next x -value. The machine connects each point to the preceding plotted point to form a representation of the graph of the equation.

EXAMPLE 1 | Choosing an Appropriate Viewing Rectangle

Graph the equation $y = x^2 + 3$ in an appropriate viewing rectangle.

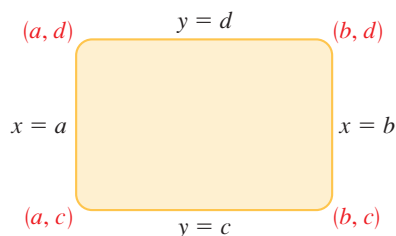


FIGURE 1 The viewing rectangle $[a, b]$ by $[c, d]$

SOLUTION Let's experiment with different viewing rectangles. We start with the viewing rectangle $[-2, 2]$ by $[-2, 2]$, so we set

$$\begin{array}{ll} X_{\min} = -2 & Y_{\min} = -2 \\ X_{\max} = 2 & Y_{\max} = 2 \end{array}$$

The resulting graph in Figure 2(a) is blank! This is because $x^2 \geq 0$, so $x^2 + 3 \geq 3$ for all x . Thus, the graph lies entirely above the viewing rectangle, so this viewing rectangle is not appropriate. If we enlarge the viewing rectangle to $[-4, 4]$ by $[-4, 4]$, as in Figure 2(b), we begin to see a portion of the graph.

Now let's try the viewing rectangle $[-10, 10]$ by $[-5, 30]$. The graph in Figure 2(c) seems to give a more complete view of the graph. If we enlarge the viewing rectangle even further, as in Figure 2(d), the graph doesn't show clearly that the y -intercept is 3.

So the viewing rectangle $[-10, 10]$ by $[-5, 30]$ gives an appropriate representation of the graph.

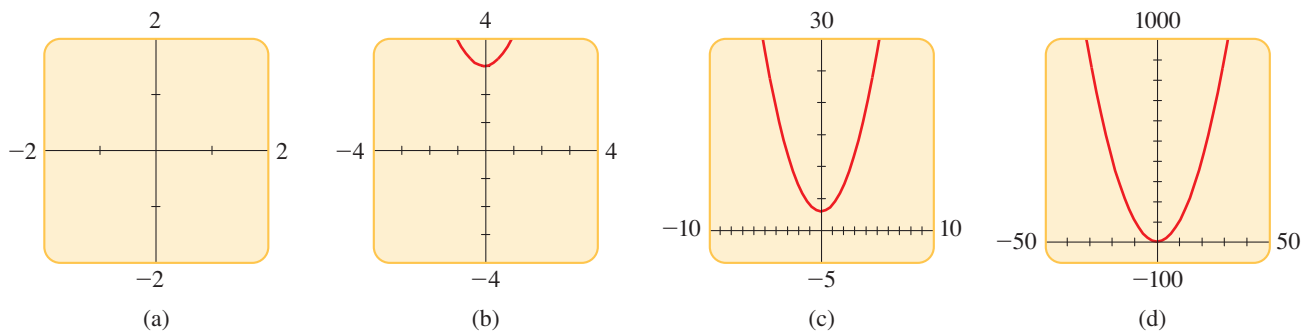


FIGURE 2 Graphs of $y = x^2 + 3$

NOW TRY EXERCISE 5

EXAMPLE 2 | Two Graphs on the Same Screen

Graph the equations $y = 3x^2 - 6x + 1$ and $y = 0.23x - 2.25$ together in the viewing rectangle $[-1, 3]$ by $[-2.5, 1.5]$. Do the graphs intersect in this viewing rectangle?

SOLUTION Figure 3(a) shows the essential features of both graphs. One is a parabola, and the other is a line. It looks as if the graphs intersect near the point $(1, -2)$. However, if we zoom in on the area around this point as shown in Figure 3(b), we see that although the graphs almost touch, they do not actually intersect.

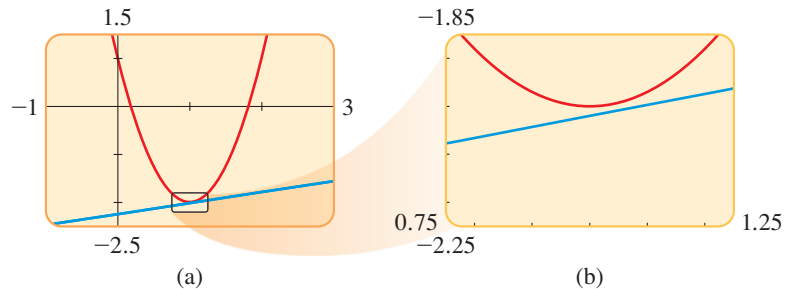


FIGURE 3

NOW TRY EXERCISE 23

You can see from Examples 1 and 2 that the choice of a viewing rectangle makes a big difference in the appearance of a graph. If you want an overview of the essential features of a graph, you must choose a relatively large viewing rectangle to obtain a global view of the graph. If you want to investigate the details of a graph, you must zoom in to a small viewing rectangle that shows just the feature of interest.

Most graphing calculators can only graph equations in which y is isolated on one side of the equal sign. The next example shows how to graph equations that don't have this property.

EXAMPLE 3 | Graphing a Circle

Graph the circle $x^2 + y^2 = 1$.

SOLUTION We first solve for y , to isolate it on one side of the equal sign.

$$y^2 = 1 - x^2 \quad \text{Subtract } x^2$$

$$y = \pm \sqrt{1 - x^2} \quad \text{Take square roots}$$

Therefore, the circle is described by the graphs of *two* equations:

$$y = \sqrt{1 - x^2} \quad \text{and} \quad y = -\sqrt{1 - x^2}$$

The first equation represents the top half of the circle (because $y \geq 0$), and the second represents the bottom half of the circle (because $y \leq 0$). If we graph the first equation in the viewing rectangle $[-2, 2]$ by $[-2, 2]$, we get the semicircle shown in Figure 4(a). The graph of the second equation is the semicircle in Figure 4(b). Graphing these semicircles together on the same viewing screen, we get the full circle in Figure 4(c).

The graph in Figure 4(c) looks somewhat flattened. Most graphing calculators allow you to set the scales on the axes so that circles really look like circles. On the TI-82 and TI-83, from the **ZOOM** menu, choose **ZSquare** to set the scales appropriately. (On the TI-86 the command is **Zsq**.)

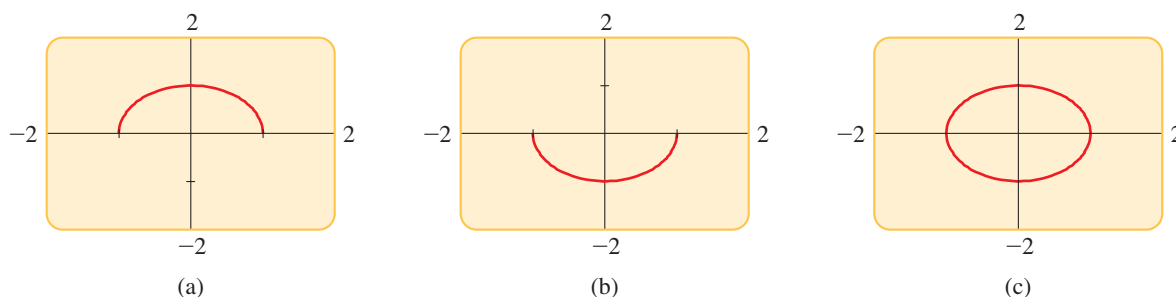


FIGURE 4 Graphing the equation $x^2 + y^2 = 1$

NOW TRY EXERCISE 27

▼ Solving Equations Graphically

In Section 1.5 we learned how to solve equations. To solve an equation such as

$$3x - 5 = 0$$

we used the **algebraic method**. This means that we used the rules of algebra to isolate x on one side of the equation. We view x as an *unknown*, and we use the rules of algebra to hunt it down. Here are the steps in the solution:

$$3x - 5 = 0$$

$$3x = 5 \quad \text{Add 5}$$

$$x = \frac{5}{3} \quad \text{Divide by 3}$$

So the solution is $x = \frac{5}{3}$.

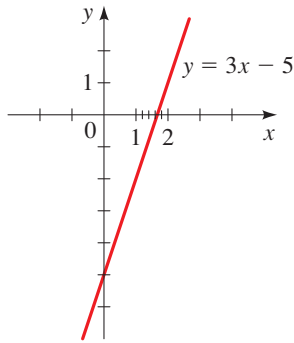


FIGURE 5

We can also solve this equation by the **graphical method**. In this method we view x as a *variable* and sketch the graph of the equation

$$y = 3x - 5$$

Different values for x give different values for y . Our goal is to find the value of x for which $y = 0$. From the graph in Figure 5 we see that $y = 0$ when $x \approx 1.7$. Thus, the solution is $x \approx 1.7$. Note that from the graph we obtain an approximate solution. We summarize these methods in the box below.

SOLVING AN EQUATION

Algebraic Method

Use the rules of algebra to isolate the unknown x on one side of the equation.

Example: $2x = 6 - x$

$$3x = 6 \quad \text{Add } x$$

$$x = 2 \quad \text{Divide by 3}$$

The solution is $x = 2$.

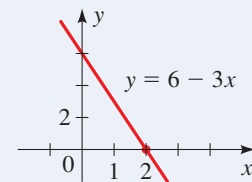
Graphical Method

Move all terms to one side and set equal to y . Sketch the graph to find the value of x where $y = 0$.

Example: $2x = 6 - x$

$$0 = 6 - 3x$$

Set $y = 6 - 3x$ and graph.



From the graph, the solution is $x \approx 2$.

The advantage of the algebraic method is that it gives exact answers. Also, the process of unraveling the equation to arrive at the answer helps us to understand the algebraic structure of the equation. On the other hand, for many equations it is difficult or impossible to isolate x .

The graphical method gives a numerical approximation to the answer. This is an advantage when a numerical answer is desired. (For example, an engineer might find an answer expressed as $x \approx 2.6$ more immediately useful than $x = \sqrt{7}$.) Also, graphing an equation helps us to visualize how the solution is related to other values of the variable.

The *Discovery Project* referenced on page 263 describes a numerical method for solving equations.



© Bettman / CORBIS

PIERRE DE FERMAT (1601–1665) was a French lawyer who became interested in mathematics at the age of 30. Because of his job as a magistrate, Fermat had little time to write complete proofs of his discoveries and often wrote them in the margin of whatever book he was reading at the time. After his death, his copy of Diophantus' *Arithmetica* (see page 20) was found to contain a particularly tantalizing comment. Where Diophantus discusses the solutions of $x^2 + y^2 = z^2$ (for example, $x = 3$, $y = 4$, and $z = 5$), Fermat

states in the margin that for $n \geq 3$ there are no natural number solutions to the equation $x^n + y^n = z^n$. In other words, it's impossible for a cube to equal the sum of two cubes, a fourth power to equal the sum of two fourth powers, and so on. Fermat writes, "I have discovered a truly wonderful proof for this but the margin is too small to contain it." All the other margin comments in Fermat's copy of *Arithmetica* have been proved. This one, however, remained unproved, and it came to be known as "Fermat's Last Theorem."

In 1994, Andrew Wiles of Princeton University announced a proof of Fermat's Last Theorem, an astounding 350 years after it was conjectured. His proof is one of the most widely reported mathematical results in the popular press.

EXAMPLE 4 Solving a Quadratic Equation Algebraically and Graphically

Solve the quadratic equations algebraically and graphically.

(a) $x^2 - 4x + 2 = 0$ (b) $x^2 - 4x + 4 = 0$ (c) $x^2 - 4x + 6 = 0$

SOLUTION 1: Algebraic

We use the Quadratic Formula to solve each equation.

(a)
$$x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4 \cdot 1 \cdot 2}}{2} = \frac{4 \pm \sqrt{8}}{2} = 2 \pm \sqrt{2}$$

There are two solutions, $x = 2 + \sqrt{2}$ and $x = 2 - \sqrt{2}$.

(b)
$$x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4 \cdot 1 \cdot 4}}{2} = \frac{4 \pm \sqrt{0}}{2} = 2$$

There is just one solution, $x = 2$.

(c)
$$x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4 \cdot 1 \cdot 6}}{2} = \frac{4 \pm \sqrt{-8}}{2}$$

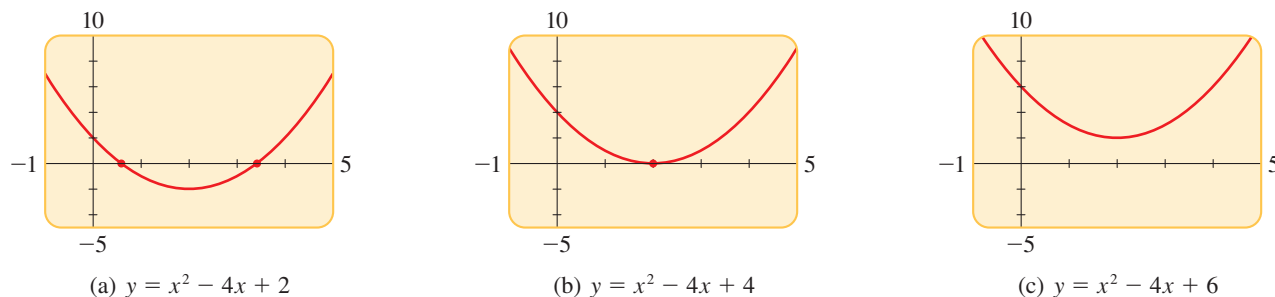
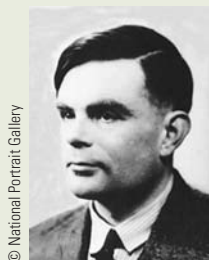
There is no real solution.

SOLUTION 2: GraphicalWe graph the equations $y = x^2 - 4x + 2$, $y = x^2 - 4x + 4$, and $y = x^2 - 4x + 6$ in Figure 6. By determining the x -intercepts of the graphs, we find the following solutions.

(a) $x \approx 0.6$ and $x \approx 3.4$

(b) $x = 2$

(c) There is no x -intercept, so the equation has no solution.

**FIGURE 6** **NOW TRY EXERCISE 35**

ALAN TURING (1912–1954) was at the center of two pivotal events of the 20th century: World War II and the invention of computers. At the age of 23 Turing made his mark on mathematics by solving an important problem in the foundations of mathematics that had been posed by David Hilbert at the 1928 International Congress of Mathematicians (see page 683). In this research he invented a theoretical machine, now called a Turing machine, which was the in-

spiration for modern digital computers. During World War II Turing was in charge of the British effort to decipher secret German codes. His complete success in this endeavor played a decisive role in the Allies' victory. To carry out the numerous logical steps that are required to break a coded message, Turing developed decision procedures similar to modern computer programs. After the war he helped to develop the first electronic computers in Britain. He also did pioneering work on artificial intelligence and computer models of biological processes. At the age of 42 Turing died of poisoning after eating an apple that had mysteriously been laced with cyanide.

The graphs in Figure 6 show visually why a quadratic equation may have two solutions, one solution, or no real solution. We proved this fact algebraically in Section 1.5 when we studied the discriminant.

EXAMPLE 5 | Another Graphical Method

Solve the equation algebraically and graphically: $5 - 3x = 8x - 20$

SOLUTION 1: Algebraic

$$5 - 3x = 8x - 20$$

$$-3x = 8x - 25 \quad \text{Subtract 5}$$

$$-11x = -25 \quad \text{Subtract } 8x$$

$$x = \frac{-25}{-11} = 2\frac{3}{11} \quad \text{Divide by } -11 \text{ and simplify}$$

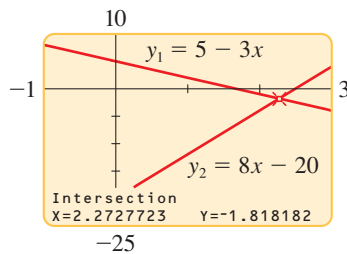


FIGURE 7

SOLUTION 2: Graphical

We could move all terms to one side of the equal sign, set the result equal to y , and graph the resulting equation. But to avoid all this algebra, we graph two equations instead:

$$y_1 = 5 - 3x \quad \text{and} \quad y_2 = 8x - 20$$

The solution of the original equation will be the value of x that makes y_1 equal to y_2 ; that is, the solution is the x -coordinate of the intersection point of the two graphs. Using the **TRACE** feature or the **intersect** command on a graphing calculator, we see from Figure 7 that the solution is $x \approx 2.27$.

NOW TRY EXERCISE 31

In the next example we use the graphical method to solve an equation that is extremely difficult to solve algebraically.

EXAMPLE 6 | Solving an Equation in an Interval

Solve the equation

$$x^3 - 6x^2 + 9x = \sqrt{x}$$

in the interval $[1, 6]$.

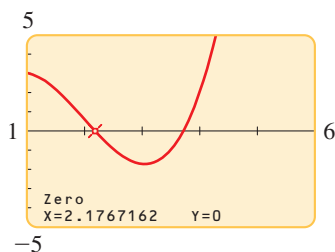
SOLUTION We are asked to find all solutions x that satisfy $1 \leq x \leq 6$, so we will graph the equation in a viewing rectangle for which the x -values are restricted to this interval.

$$x^3 - 6x^2 + 9x = \sqrt{x}$$

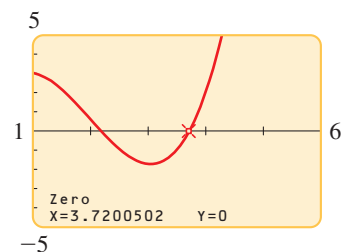
$$x^3 - 6x^2 + 9x - \sqrt{x} = 0 \quad \text{Subtract } \sqrt{x}$$

We can also use the **zero** command to find the solutions, as shown in Figures 8(a) and 8(b).

Figure 8 shows the graph of the equation $y = x^3 - 6x^2 + 9x - \sqrt{x}$ in the viewing rectangle $[1, 6]$ by $[-5, 5]$. There are two x -intercepts in this viewing rectangle; zooming in, we see that the solutions are $x \approx 2.18$ and $x \approx 3.72$.



(a)



(b)

FIGURE 8

NOW TRY EXERCISE 43

The equation in Example 6 actually has four solutions. You are asked to find the other two in Exercise 71.

EXAMPLE 7 | Intensity of Light

Two light sources are 10 m apart. One is three times as intense as the other. The light intensity L (in lux) at a point x meters from the weaker source is given by

$$L = \frac{10}{x^2} + \frac{30}{(10 - x)^2}$$

(See Figure 9.) Find the points at which the light intensity is 4 lux.

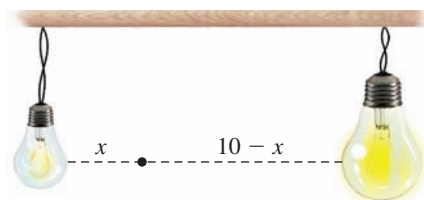


FIGURE 9

SOLUTION We need to solve the equation

$$4 = \frac{10}{x^2} + \frac{30}{(10 - x)^2}$$

The graphs of

$$y_1 = 4 \quad \text{and} \quad y_2 = \frac{10}{x^2} + \frac{30}{(10 - x)^2}$$

are shown in Figure 10. Zooming in (or using the `intersect` command), we find two solutions, $x \approx 1.67431$ and $x \approx 7.1927193$. So the light intensity is 4 lux at the points that are 1.67 m and 7.19 m from the weaker source.

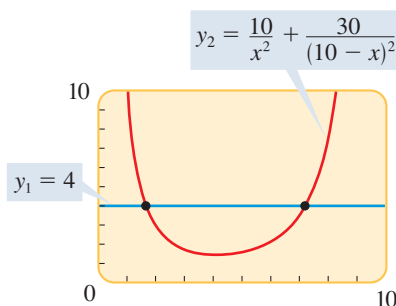


FIGURE 10

 **NOW TRY EXERCISE 73**

▼ Solving Inequalities Graphically

Inequalities can be solved graphically. To describe the method, we solve

$$x^2 - 5x + 6 \leq 0$$

This inequality was solved algebraically in Example 3 of Section 1.7. To solve the inequality graphically, we draw the graph of

$$y = x^2 - 5x + 6$$

Our goal is to find those values of x for which $y \leq 0$. These are simply the x -values for which the graph lies below the x -axis. From Figure 11 we see that the solution of the inequality is the interval $[2, 3]$.

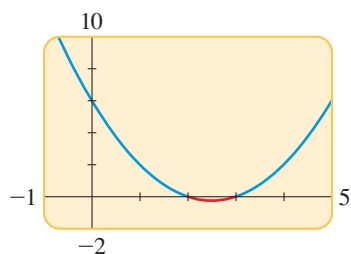


FIGURE 11 $x^2 - 5x + 6 \leq 0$

EXAMPLE 8 | Solving an Inequality Graphically

Solve the inequality $3.7x^2 + 1.3x - 1.9 \leq 2.0 - 1.4x$.

SOLUTION We graph the equations

$$y_1 = 3.7x^2 + 1.3x - 1.9 \quad \text{and} \quad y_2 = 2.0 - 1.4x$$

in the same viewing rectangle in Figure 12. We are interested in those values of x for which $y_1 \leq y_2$; these are points for which the graph of y_2 lies on or above the graph of y_1 . To determine the appropriate interval, we look for the x -coordinates of points where the graphs intersect. We conclude that the solution is (approximately) the interval $[-1.45, 0.72]$.

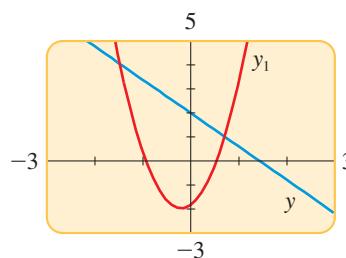


FIGURE 12

$$y_1 = 3.7x^2 + 1.3x - 1.9$$

$$y_2 = 2.0 - 1.4x$$

 **NOW TRY EXERCISE 59**

EXAMPLE 9 | Solving an Inequality Graphically

Solve the inequality $x^3 - 5x^2 \geq -8$.

SOLUTION We write the inequality as

$$x^3 - 5x^2 + 8 \geq 0$$

and then graph the equation

$$y = x^3 - 5x^2 + 8$$

in the viewing rectangle $[-6, 6]$ by $[-15, 15]$, as shown in Figure 13. The solution of the inequality consists of those intervals on which the graph lies on or above the x -axis. By moving the cursor to the x -intercepts, we find that, rounded to one decimal place, the solution is $[-1.1, 1.5] \cup [4.6, \infty)$.

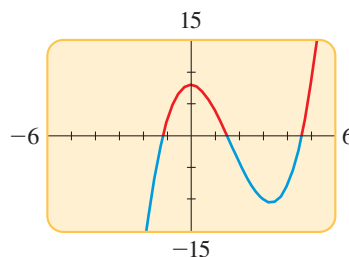


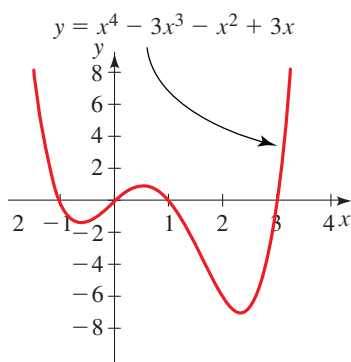
FIGURE 13 $x^3 - 5x^2 + 8 \geq 0$

 **NOW TRY EXERCISE 61**

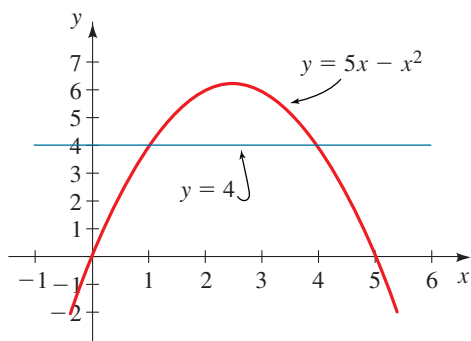
1.9 EXERCISES

CONCEPTS

- The solutions of the equation $x^2 - 2x - 3 = 0$ are the _____-intercepts of the graph of $y = x^2 - 2x - 3$.
- The solutions of the inequality $x^2 - 2x - 3 > 0$ are the x -coordinates of the points on the graph of $y = x^2 - 2x - 3$ that lie _____ the x -axis.
- The figure shows a graph of $y = x^4 - 3x^3 - x^2 + 3x$. Use the graph to do the following.
 - Find the solutions of the equation $x^4 - 3x^3 - x^2 + 3x = 0$.
 - Find the solutions of the inequality $x^4 - 3x^3 - x^2 + 3x \leq 0$.



- The figure shows the graphs of $y = 5x - x^2$ and $y = 4$. Use the graphs to do the following.
 - Find the solutions of the equation $5x - x^2 = 4$.
 - Find the solutions of the inequality $5x - x^2 > 4$.



SKILLS

5–10 ■ Use a graphing calculator or computer to decide which viewing rectangle (a)–(d) produces the most appropriate graph of the equation.

- $y = x^4 + 2$
 - $[-2, 2]$ by $[-2, 2]$
 - $[0, 4]$ by $[0, 4]$
 - $[-8, 8]$ by $[-4, 40]$
 - $[-40, 40]$ by $[-80, 800]$

- $y = x^2 + 7x + 6$
 - $[-5, 5]$ by $[-5, 5]$
 - $[0, 10]$ by $[-20, 100]$
 - $[-15, 8]$ by $[-20, 100]$
 - $[-10, 3]$ by $[-100, 20]$

- $y = 100 - x^2$
 - $[-4, 4]$ by $[-4, 4]$
 - $[-10, 10]$ by $[-10, 10]$
 - $[-15, 15]$ by $[-30, 110]$
 - $[-4, 4]$ by $[-30, 110]$

- $y = 2x^2 - 1000$
 - $[-10, 10]$ by $[-10, 10]$
 - $[-10, 10]$ by $[-100, 100]$
 - $[-10, 10]$ by $[-1000, 1000]$
 - $[-25, 25]$ by $[-1200, 200]$

- $y = 10 + 25x - x^3$
 - $[-4, 4]$ by $[-4, 4]$
 - $[-10, 10]$ by $[-10, 10]$
 - $[-20, 20]$ by $[-100, 100]$
 - $[-100, 100]$ by $[-200, 200]$

- $y = \sqrt{8x - x^2}$
 - $[-4, 4]$ by $[-4, 4]$
 - $[-5, 5]$ by $[0, 100]$
 - $[-10, 10]$ by $[-10, 40]$
 - $[-2, 10]$ by $[-2, 6]$

11–22 ■ Determine an appropriate viewing rectangle for the equation, and use it to draw the graph.

- | | |
|-------------------------------|------------------------------|
| 11. $y = 100x^2$ | 12. $y = -100x^2$ |
| 13. $y = 4 + 6x - x^2$ | 14. $y = 0.3x^2 + 1.7x - 3$ |
| 15. $y = \sqrt[4]{256 - x^2}$ | 16. $y = \sqrt{12x - 17}$ |
| 17. $y = 0.01x^3 - x^2 + 5$ | 18. $y = x(x + 6)(x - 9)$ |
| 19. $y = x^4 - 4x^3$ | 20. $y = \frac{x}{x^2 + 25}$ |
| 21. $y = 1 + x - 1 $ | 22. $y = 2x - x^2 - 5 $ |

23–26 ■ Do the graphs intersect in the given viewing rectangle? If they do, how many points of intersection are there?

- $y = -3x^2 + 6x - \frac{1}{2}$, $y = \sqrt{7 - \frac{7}{12}x^2}$; $[-4, 4]$ by $[-1, 3]$
- $y = \sqrt{49 - x^2}$, $y = \frac{1}{5}(41 - 3x)$; $[-8, 8]$ by $[-1, 8]$
- $y = 6 - 4x - x^2$, $y = 3x + 18$; $[-6, 2]$ by $[-5, 20]$
- $y = x^3 - 4x$, $y = x + 5$; $[-4, 4]$ by $[-15, 15]$
- Graph the circle $x^2 + y^2 = 9$ by solving for y and graphing two equations as in Example 3.
- Graph the circle $(y - 1)^2 + x^2 = 1$ by solving for y and graphing two equations as in Example 3.
- Graph the equation $4x^2 + 2y^2 = 1$ by solving for y and graphing two equations corresponding to the negative and positive square roots. (This graph is called an *ellipse*.)

30. Graph the equation $y^2 - 9x^2 = 1$ by solving for y and graphing the two equations corresponding to the positive and negative square roots. (This graph is called a *hyperbola*.)

31–42 ■ Solve the equation both algebraically and graphically.

31. $x - 4 = 5x + 12$ 32. $\frac{1}{2}x - 3 = 6 + 2x$
 33. $\frac{2}{x} + \frac{1}{2x} = 7$ 34. $\frac{4}{x+2} - \frac{6}{2x} = \frac{5}{2x+4}$
 35. $x^2 - 32 = 0$ 36. $x^3 + 16 = 0$
 37. $x^2 + 9 = 0$ 38. $x^2 + 3 = 2x$
 39. $16x^4 = 625$ 40. $2x^5 - 243 = 0$
 41. $(x - 5)^4 - 80 = 0$ 42. $6(x + 2)^5 = 64$

43–50 ■ Solve the equation graphically in the given interval. State each answer rounded to two decimals.

43. $x^2 - 7x + 12 = 0$; $[0, 6]$
 44. $x^2 - 0.75x + 0.125 = 0$; $[-2, 2]$
 45. $x^3 - 6x^2 + 11x - 6 = 0$; $[-1, 4]$
 46. $16x^3 + 16x^2 = x + 1$; $[-2, 2]$
 47. $x - \sqrt{x+1} = 0$; $[-1, 5]$
 48. $1 + \sqrt{x} = \sqrt{1+x^2}$; $[-1, 5]$
 49. $x^{1/3} - x = 0$; $[-3, 3]$
 50. $x^{1/2} + x^{1/3} - x = 0$; $[-1, 5]$

51–54 ■ Use the graphical method to solve the equation in the indicated exercise from Section 1.5.

51. Exercise 91 52. Exercise 92
 53. Exercise 97 54. Exercise 98

55–58 ■ Find all real solutions of the equation, rounded to two decimals.

55. $x^3 - 2x^2 - x - 1 = 0$ 56. $x^4 - 8x^2 + 2 = 0$
 57. $x(x-1)(x+2) = \frac{1}{6}x$ 58. $x^4 = 16 - x^3$

59–66 ■ Find the solutions of the inequality by drawing appropriate graphs. State each answer rounded to two decimals.

59. $x^2 \leq 3x + 10$ 60. $0.5x^2 + 0.875x \leq 0.25$
 61. $x^3 + 11x \leq 6x^2 + 6$ 62. $16x^3 + 24x^2 > -9x - 1$
 63. $x^{1/3} < x$ 64. $\sqrt{0.5x^2 + 1} \leq 2|x|$
 65. $(x+1)^2 < (x-1)^2$ 66. $(x+1)^2 \leq x^3$

67–70 ■ Use the graphical method to solve the inequality in the indicated exercise from Section 1.7.

67. Exercise 43 68. Exercise 44
 69. Exercise 53 70. Exercise 54

71. In Example 6 we found two solutions of the equation $x^3 - 6x^2 + 9x = \sqrt{x}$, the solutions that lie between 1 and 6. Find two more solutions, correct to two decimals.

APPLICATIONS

72. **Estimating Profit** An appliance manufacturer estimates that the profit y (in dollars) generated by producing x cooktops per month is given by the equation

$$y = 10x + 0.5x^2 - 0.001x^3 - 5000$$

where $0 \leq x \leq 450$.

- (a) Graph the equation.
 (b) How many cooktops must be produced to begin generating a profit?
 (c) For what range of values of x is the company's profit greater than \$15,000?

73. **How Far Can You See?** If you stand on a ship in a calm sea, then your height x (in ft) above sea level is related to the farthest distance y (in mi) that you can see by the equation

$$y = \sqrt{1.5x + \left(\frac{x}{5280}\right)^2}$$

- (a) Graph the equation for $0 \leq x \leq 100$.
 (b) How high up do you have to be to be able to see 10 mi?



DISCOVERY ■ DISCUSSION ■ WRITING

74. **Misleading Graphs** Write a short essay describing different ways in which a graphing calculator might give a misleading graph of an equation.
75. **Algebraic and Graphical Solution Methods** Write a short essay comparing the algebraic and graphical methods for solving equations. Make up your own examples to illustrate the advantages and disadvantages of each method.
76. **Equation Notation on Graphing Calculators** When you enter the following equations into your calculator, how does what you see on the screen differ from the usual way of writing the equations? (Check your user's manual if you're not sure.)
 (a) $y = |x|$ (b) $y = \sqrt[3]{x}$
 (c) $y = \frac{x}{x-1}$ (d) $y = x^3 + \sqrt[3]{x+2}$
77. **Enter Equations Carefully** A student wishes to graph the equations

$$y = x^{1/3} \quad \text{and} \quad y = \frac{x}{x+4}$$

on the same screen, so he enters the following information into his calculator:

$$Y_1 = X^{1/3} \quad Y_2 = X/X+4$$

The calculator graphs two lines instead of the equations he wanted. What went wrong?

1.10 LINES

The Slope of a Line ► Point-Slope Form of the Equation of a Line ► Slope-Intercept Form of the Equation of a Line ► Vertical and Horizontal Lines ► General Equation of a Line ► Parallel and Perpendicular Lines ► Modeling with Linear Equations: Slope as Rate of Change

In this section we find equations for straight lines lying in a coordinate plane. The equations will depend on how the line is inclined, so we begin by discussing the concept of slope.

▼ The Slope of a Line

We first need a way to measure the “steepness” of a line, or how quickly it rises (or falls) as we move from left to right. We define *run* to be the distance we move to the right and *rise* to be the corresponding distance that the line rises (or falls). The *slope* of a line is the ratio of rise to run:

$$\text{slope} = \frac{\text{rise}}{\text{run}}$$

Figure 1 shows situations in which slope is important. Carpenters use the term *pitch* for the slope of a roof or a staircase; the term *grade* is used for the slope of a road.

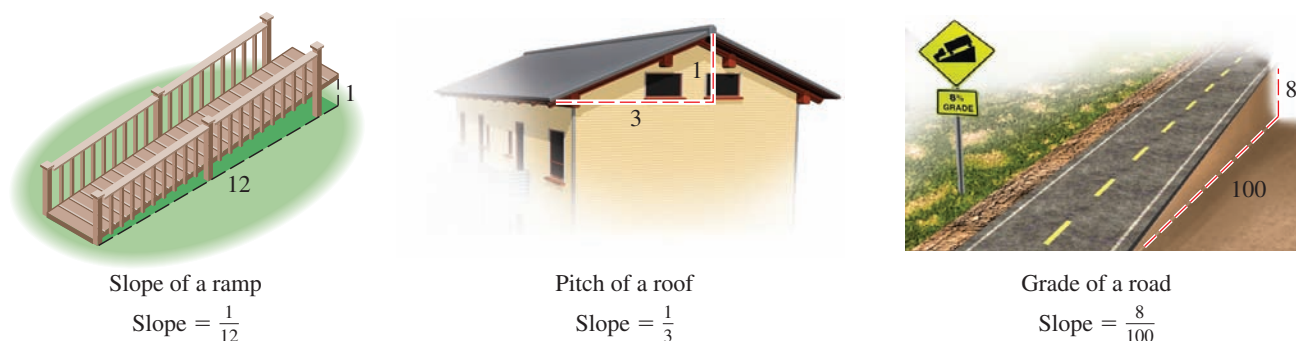


FIGURE 1

If a line lies in a coordinate plane, then the **run** is the change in the x -coordinate and the **rise** is the corresponding change in the y -coordinate between any two points on the line (see Figure 2). This gives us the following definition of slope.

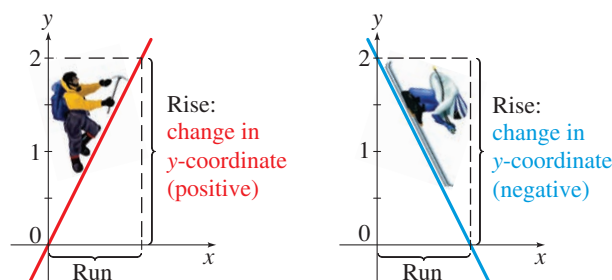


FIGURE 2

SLOPE OF A LINE

The **slope** m of a nonvertical line that passes through the points $A(x_1, y_1)$ and $B(x_2, y_2)$ is

$$m = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1}$$

The slope of a vertical line is not defined.

The slope is independent of which two points are chosen on the line. We can see that this is true from the similar triangles in Figure 3:

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{y'_2 - y'_1}{x'_2 - x'_1}$$

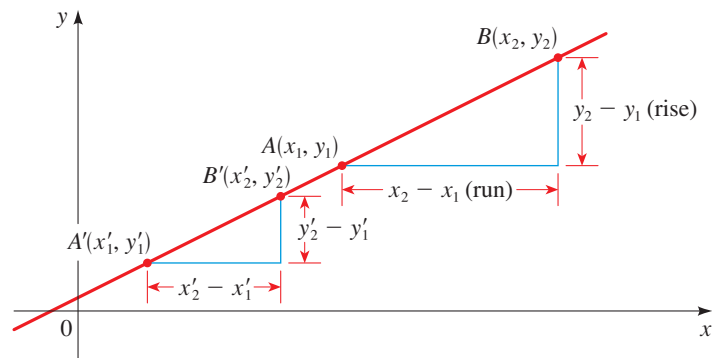
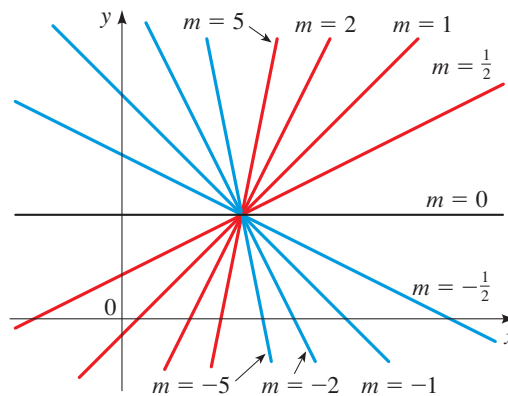
**FIGURE 3**

Figure 4 shows several lines labeled with their slopes. Notice that lines with positive slope slant upward to the right, whereas lines with negative slope slant downward to the right. The steepest lines are those for which the absolute value of the slope is the largest; a horizontal line has slope zero.

**FIGURE 4** Lines with various slopes**EXAMPLE 1** | Finding the Slope of a Line Through Two Points

Find the slope of the line that passes through the points $P(2, 1)$ and $Q(8, 5)$.

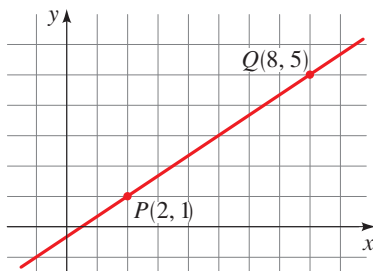


FIGURE 5

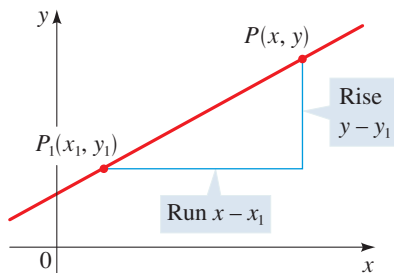


FIGURE 6

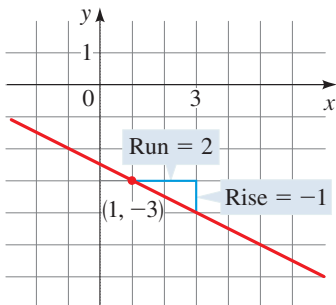


FIGURE 7

SOLUTION Since any two different points determine a line, only one line passes through these two points. From the definition the slope is

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{5 - 1}{8 - 2} = \frac{4}{6} = \frac{2}{3}$$

This says that for every 3 units we move to the right, the line rises 2 units. The line is drawn in Figure 5.

NOW TRY EXERCISE 5

Point-Slope Form of the Equation of a Line

Now let's find the equation of the line that passes through a given point $P(x_1, y_1)$ and has slope m . A point $P(x, y)$ with $x \neq x_1$ lies on this line if and only if the slope of the line through P_1 and P is equal to m (see Figure 6), that is,

$$\frac{y - y_1}{x - x_1} = m$$

This equation can be rewritten in the form $y - y_1 = m(x - x_1)$; note that the equation is also satisfied when $x = x_1$ and $y = y_1$. Therefore, it is an equation of the given line.

POINT-SLOPE FORM OF THE EQUATION OF A LINE

An equation of the line that passes through the point (x_1, y_1) and has slope m is

$$y - y_1 = m(x - x_1)$$

EXAMPLE 2 Finding the Equation of a Line with Given Point and Slope

- Find an equation of the line through $(1, -3)$ with slope $-\frac{1}{2}$.
- Sketch the line.

SOLUTION

- Using the point-slope form with $m = -\frac{1}{2}$, $x_1 = 1$, and $y_1 = -3$, we obtain an equation of the line as

$$y + 3 = -\frac{1}{2}(x - 1) \quad \text{Slope } m = -\frac{1}{2}, \text{ point } (1, -3)$$

$$2y + 6 = -x + 1 \quad \text{Multiply by 2}$$

$$x + 2y + 5 = 0 \quad \text{Rearrange}$$

- The fact that the slope is $-\frac{1}{2}$ tells us that when we move to the right 2 units, the line drops 1 unit. This enables us to sketch the line in Figure 7.

NOW TRY EXERCISE 19

EXAMPLE 3 Finding the Equation of a Line Through Two Given Points

Find an equation of the line through the points $(-1, 2)$ and $(3, -4)$.

SOLUTION The slope of the line is

$$m = \frac{-4 - 2}{3 - (-1)} = -\frac{6}{4} = -\frac{3}{2}$$

We can use *either* point, $(-1, 2)$ or $(3, -4)$, in the point-slope equation. We will end up with the same final answer.

Using the point-slope form with $x_1 = -1$ and $y_1 = 2$, we obtain

$$y - 2 = -\frac{3}{2}(x + 1) \quad \text{Slope } m = -\frac{3}{2}, \text{ point } (-1, 2)$$

$$2y - 4 = -3x - 3 \quad \text{Multiply by 2}$$

$$3x + 2y - 1 = 0 \quad \text{Rearrange}$$

 **NOW TRY EXERCISE 23**

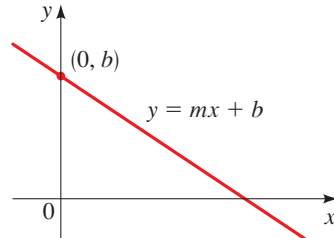


FIGURE 8

▼ Slope-Intercept Form of the Equation of a Line

Suppose a nonvertical line has slope m and y -intercept b (see Figure 8). This means that the line intersects the y -axis at the point $(0, b)$, so the point-slope form of the equation of the line, with $x = 0$ and $y = b$, becomes

$$y - b = m(x - 0)$$

This simplifies to $y = mx + b$, which is called the **slope-intercept form** of the equation of a line.

SLOPE-INTERCEPT FORM OF THE EQUATION OF A LINE

An equation of the line that has slope m and y -intercept b is

$$y = mx + b$$

EXAMPLE 4 | Lines in Slope-Intercept Form

- (a) Find the equation of the line with slope 3 and y -intercept -2 .
- (b) Find the slope and y -intercept of the line $3y - 2x = 1$.

SOLUTION

- (a) Since $m = 3$ and $b = -2$, from the slope-intercept form of the equation of a line we get

$$y = 3x - 2$$

- (b) We first write the equation in the form $y = mx + b$:

$$3y - 2x = 1$$

$$3y = 2x + 1 \quad \text{Add } 2x$$

$$y = \frac{2}{3}x + \frac{1}{3} \quad \text{Divide by 3}$$

From the slope-intercept form of the equation of a line, we see that the slope is $m = \frac{2}{3}$ and the y -intercept is $b = \frac{1}{3}$.

 **NOW TRY EXERCISES 25 AND 47**

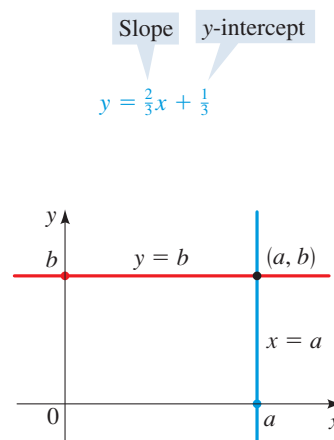


FIGURE 9

▼ Vertical and Horizontal Lines

If a line is horizontal, its slope is $m = 0$, so its equation is $y = b$, where b is the y -intercept (see Figure 9). A vertical line does not have a slope, but we can write its equation as $x = a$, where a is the x -intercept, because the x -coordinate of every point on the line is a .

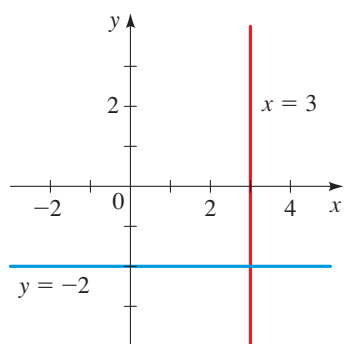


FIGURE 10

VERTICAL AND HORIZONTAL LINES

An equation of the vertical line through (a, b) is $x = a$.

An equation of the horizontal line through (a, b) is $y = b$.

EXAMPLE 5 | Vertical and Horizontal Lines

- (a) An equation for the vertical line through $(3, 5)$ is $x = 3$.
- (b) The graph of the equation $x = 3$ is a vertical line with x -intercept 3.
- (c) An equation for the horizontal line through $(8, -2)$ is $y = -2$.
- (d) The graph of the equation $y = -2$ is a horizontal line with y -intercept -2 .

The lines are graphed in Figure 10.

 **NOW TRY EXERCISES 29 AND 33**

General Equation of a Line

A **linear equation** is an equation of the form

$$Ax + By + C = 0$$

where A , B , and C are constants and A and B are not both 0. The equation of a line is a linear equation:

- A nonvertical line has the equation $y = mx + b$ or $-mx + y - b = 0$, which is a linear equation with $A = -m$, $B = 1$, and $C = -b$.
- A vertical line has the equation $x = a$ or $x - a = 0$, which is a linear equation with $A = 1$, $B = 0$, and $C = -a$.

Conversely, the graph of a linear equation is a line.

- If $B \neq 0$, the equation becomes

$$y = -\frac{A}{B}x - \frac{C}{B} \quad \text{Divide by } B$$

and this is the slope-intercept form of the equation of a line (with $m = -A/B$ and $b = -C/B$).

- If $B = 0$, the equation becomes

$$Ax + C = 0 \quad \text{Set } B = 0$$

or $x = -C/A$, which represents a vertical line.

We have proved the following.

GENERAL EQUATION OF A LINE

The graph of every **linear equation**

$$Ax + By + C = 0 \quad (A, B \text{ not both zero})$$

is a line. Conversely, every line is the graph of a linear equation.

EXAMPLE 6 | Graphing a Linear Equation

Sketch the graph of the equation $2x - 3y - 12 = 0$.

SOLUTION 1 Since the equation is linear, its graph is a line. To draw the graph, it is enough to find any two points on the line. The intercepts are the easiest points to find.

x -intercept: Substitute $y = 0$, to get $2x - 12 = 0$, so $x = 6$

y -intercept: Substitute $x = 0$, to get $-3y - 12 = 0$, so $y = -4$

With these points we can sketch the graph in Figure 11.

SOLUTION 2 We write the equation in slope-intercept form:

$$2x - 3y - 12 = 0$$

$$2x - 3y = 12 \quad \text{Add 12}$$

$$-3y = -2x + 12 \quad \text{Subtract } 2x$$

$$y = \frac{2}{3}x - 4 \quad \text{Divide by } -3$$

This equation is in the form $y = mx + b$, so the slope is $m = \frac{2}{3}$ and the y -intercept is $b = -4$. To sketch the graph, we plot the y -intercept and then move 3 units to the right and 2 units up as shown in Figure 12.

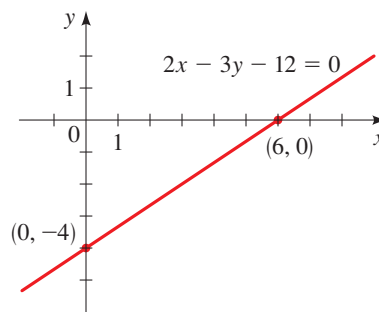


FIGURE 11

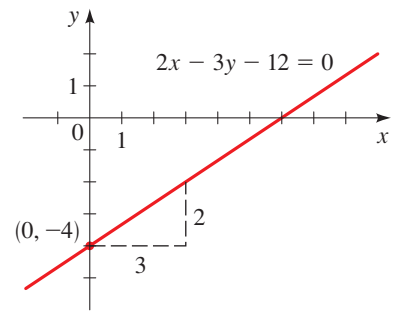


FIGURE 12

 NOW TRY EXERCISE 53

Parallel and Perpendicular Lines

Since slope measures the steepness of a line, it seems reasonable that parallel lines should have the same slope. In fact, we can prove this.

PARALLEL LINES

Two nonvertical lines are parallel if and only if they have the same slope.

PROOF Let the lines l_1 and l_2 in Figure 13 have slopes m_1 and m_2 . If the lines are parallel, then the right triangles ABC and DEF are similar, so

$$m_1 = \frac{d(B, C)}{d(A, C)} = \frac{d(E, F)}{d(D, F)} = m_2$$

Conversely, if the slopes are equal, then the triangles will be similar, so $\angle BAC = \angle EDF$ and the lines are parallel.

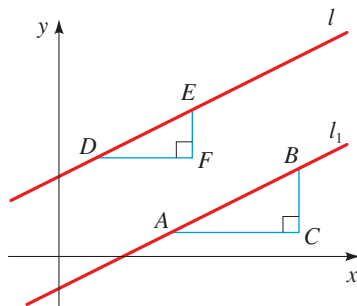


FIGURE 13

EXAMPLE 7 | Finding the Equation of a Line Parallel to a Given Line

Find an equation of the line through the point $(5, 2)$ that is parallel to the line $4x + 6y + 5 = 0$.

SOLUTION First we write the equation of the given line in slope-intercept form.

$$4x + 6y + 5 = 0$$

$$6y = -4x - 5 \quad \text{Subtract } 4x + 5$$

$$y = -\frac{2}{3}x - \frac{5}{6} \quad \text{Divide by 6}$$

So the line has slope $m = -\frac{2}{3}$. Since the required line is parallel to the given line, it also has slope $m = -\frac{2}{3}$. From the point-slope form of the equation of a line, we get

$$y - 2 = -\frac{2}{3}(x - 5) \quad \text{Slope } m = -\frac{2}{3}, \text{ point } (5, 2)$$

$$3y - 6 = -2x + 10 \quad \text{Multiply by 3}$$

$$2x + 3y - 16 = 0 \quad \text{Rearrange}$$

Thus, the equation of the required line is $2x + 3y - 16 = 0$.

 NOW TRY EXERCISE 31

The condition for perpendicular lines is not as obvious as that for parallel lines.

PERPENDICULAR LINES

Two lines with slopes m_1 and m_2 are perpendicular if and only if $m_1 m_2 = -1$, that is, their slopes are negative reciprocals:

$$m_2 = -\frac{1}{m_1}$$

Also, a horizontal line (slope 0) is perpendicular to a vertical line (no slope).

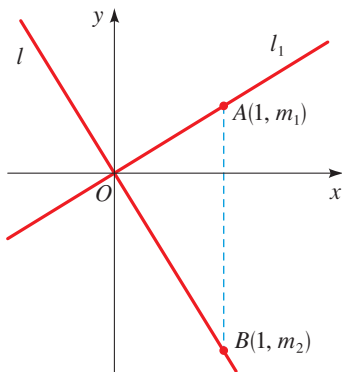


FIGURE 14

PROOF In Figure 14 we show two lines intersecting at the origin. (If the lines intersect at some other point, we consider lines parallel to these that intersect at the origin. These lines have the same slopes as the original lines.)

If the lines l_1 and l_2 have slopes m_1 and m_2 , then their equations are $y = m_1 x$ and $y = m_2 x$. Notice that $A(1, m_1)$ lies on l_1 and $B(1, m_2)$ lies on l_2 . By the Pythagorean Theorem and its converse (see page 219), $OA \perp OB$ if and only if

$$[d(O, A)]^2 + [d(O, B)]^2 = [d(A, B)]^2$$

By the Distance Formula this becomes

$$(1^2 + m_1^2) + (1^2 + m_2^2) = (1 - 1)^2 + (m_2 - m_1)^2$$

$$2 + m_1^2 + m_2^2 = m_2^2 - 2m_1 m_2 + m_1^2$$

$$2 = -2m_1 m_2$$

$$m_1 m_2 = -1$$

EXAMPLE 8 | Perpendicular Lines

Show that the points $P(3, 3)$, $Q(8, 17)$, and $R(11, 5)$ are the vertices of a right triangle.

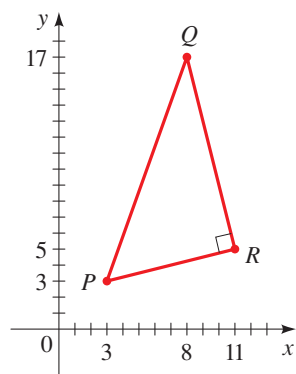


FIGURE 15

SOLUTION The slopes of the lines containing PR and QR are, respectively,

$$m_1 = \frac{5 - 3}{11 - 3} = \frac{1}{4} \quad \text{and} \quad m_2 = \frac{5 - 17}{11 - 8} = -4$$

Since $m_1 m_2 = -1$, these lines are perpendicular, so PQR is a right triangle. It is sketched in Figure 15.

 **NOW TRY EXERCISE 57**

EXAMPLE 9 | Finding an Equation of a Line Perpendicular to a Given Line

Find an equation of the line that is perpendicular to the line $4x + 6y + 5 = 0$ and passes through the origin.

SOLUTION In Example 7 we found that the slope of the line $4x + 6y + 5 = 0$ is $-\frac{2}{3}$. Thus, the slope of a perpendicular line is the negative reciprocal, that is, $\frac{3}{2}$. Since the required line passes through $(0, 0)$, the point-slope form gives

$$\begin{aligned} y - 0 &= \frac{3}{2}(x - 0) && \text{Slope } m = \frac{3}{2}, \text{ point } (0, 0) \\ y &= \frac{3}{2}x && \text{Simplify} \end{aligned}$$

 **NOW TRY EXERCISE 35**

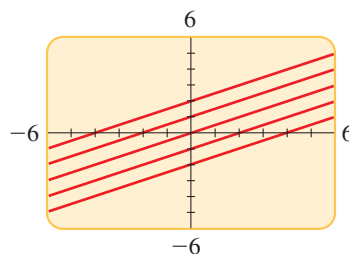
EXAMPLE 10 | Graphing a Family of Lines

Use a graphing calculator to graph the family of lines

$$y = 0.5x + b$$

for $b = -2, -1, 0, 1, 2$. What property do the lines share?

SOLUTION The lines are graphed in Figure 16 in the viewing rectangle $[-6, 6]$ by $[-6, 6]$. The lines all have the same slope, so they are parallel.

FIGURE 16 $y = 0.5x + b$

 **NOW TRY EXERCISE 41**

▼ Modeling with Linear Equations: Slope as Rate of Change

When a line is used to model the relationship between two quantities, the slope of the line is the **rate of change** of one quantity with respect to the other. For example, the graph in Figure 17(a) on the next page gives the amount of gas in a tank that is being filled. The slope between the indicated points is

$$m = \frac{6 \text{ gallons}}{3 \text{ minutes}} = 2 \text{ gal/min}$$

The slope is the *rate* at which the tank is being filled, 2 gallons per minute. In Figure 17(b) the tank is being drained at the *rate* of 0.03 gallon per minute, and the slope is -0.03 .

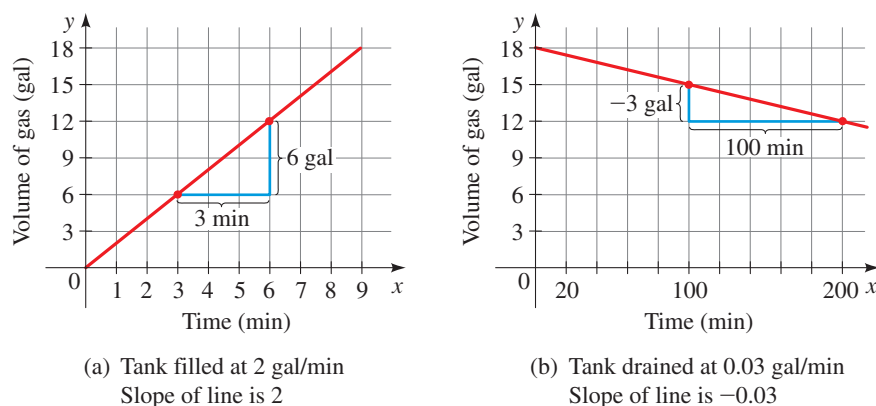


FIGURE 17

The next two examples give other situations in which the slope of a line is a rate of change.

EXAMPLE 11 | Slope as Rate of Change

A dam is built on a river to create a reservoir. The water level w in the reservoir is given by the equation

$$w = 4.5t + 28$$

where t is the number of years since the dam was constructed and w is measured in feet.

- Sketch a graph of this equation.
- What do the slope and w -intercept of this graph represent?

SOLUTION

- This equation is linear, so its graph is a line. Since two points determine a line, we plot two points that lie on the graph and draw a line through them.

When $t = 0$, then $w = 4.5(0) + 28 = 28$, so $(0, 28)$ is on the line.

When $t = 2$, then $w = 4.5(2) + 28 = 37$, so $(2, 37)$ is on the line.

The line that is determined by these points is shown in Figure 18.

- The slope is $m = 4.5$; it represents the rate of change of water level with respect to time. This means that the water level *increases* 4.5 ft per year. The w -intercept is 28 and occurs when $t = 0$, so it represents the water level when the dam was constructed.

NOW TRY EXERCISE 69

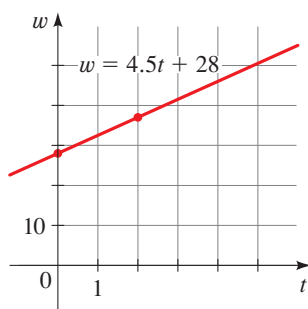


FIGURE 18

EXAMPLE 12 | Linear Relationship Between Temperature and Elevation

- As dry air moves upward, it expands and cools. If the ground temperature is 20°C and the temperature at a height of 1 km is 10°C , express the temperature T (in $^\circ\text{C}$) in terms of the height h (in km). (Assume that the relationship between T and h is linear.)

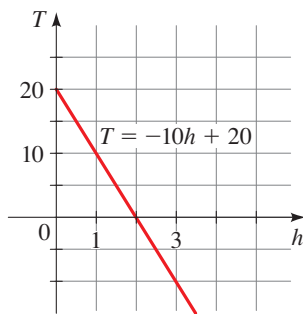


FIGURE 19

- (b) Draw the graph of the linear equation. What does its slope represent?
 (c) What is the temperature at a height of 2.5 km?

SOLUTION

- (a) Because we are assuming a linear relationship between T and h , the equation must be of the form

$$T = mh + b$$

where m and b are constants. When $h = 0$, we are given that $T = 20$, so

$$20 = m(0) + b$$

$$b = 20$$

Thus, we have

$$T = mh + 20$$

When $h = 1$, we have $T = 10$, so

$$10 = m(1) + 20$$

$$m = 10 - 20 = -10$$

The required expression is

$$T = -10h + 20$$

- (b) The graph is sketched in Figure 19. The slope is $m = -10^\circ\text{C}/\text{km}$, and this represents the rate of change of temperature with respect to distance above the ground. So the temperature *decreases* 10°C per kilometer of height.
 (c) At a height of $h = 2.5$ km the temperature is

$$T = -10(2.5) + 20 = -25 + 20 = -5^\circ\text{C}$$

NOW TRY EXERCISE 73

1.10 EXERCISES

CONCEPTS

- We find the “steepness,” or slope, of a line passing through two points by dividing the difference in the ____-coordinates of these points by the difference in the ____-coordinates. So the line passing through the points $(0, 1)$ and $(2, 5)$ has slope ____.
- A line has the equation $y = 3x + 2$.
 - This line has slope ____.
 - Any line parallel to this line has slope ____.
 - Any line perpendicular to this line has slope ____.
- The point-slope form of the equation of the line with slope 3 passing through the point $(1, 2)$ is ____.

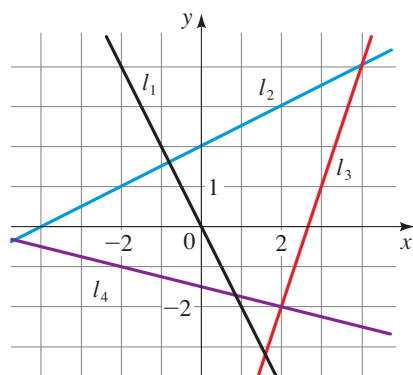
- The slope of a horizontal line is _____. The equation of the horizontal line passing through $(2, 3)$ is _____.
- The slope of a vertical line is _____. The equation of the vertical line passing through $(2, 3)$ is _____.

SKILLS

5–12 ■ Find the slope of the line through P and Q .

- | | |
|--------------------------|--------------------------|
| 5. $P(0, 0), Q(4, 2)$ | 6. $P(0, 0), Q(2, -6)$ |
| 7. $P(2, 2), Q(-10, 0)$ | 8. $P(1, 2), Q(3, 3)$ |
| 9. $P(2, 4), Q(4, 3)$ | 10. $P(2, -5), Q(-4, 3)$ |
| 11. $P(1, -3), Q(-1, 6)$ | 12. $P(-1, -4), Q(6, 0)$ |

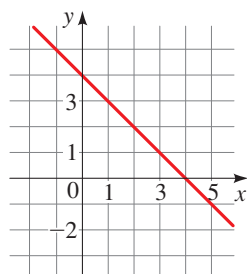
13. Find the slopes of the lines l_1 , l_2 , l_3 , and l_4 in the figure below.



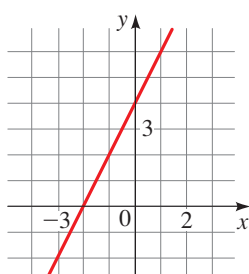
14. (a) Sketch lines through $(0, 0)$ with slopes $1, 0, \frac{1}{2}, 2,$ and -1 .
 (b) Sketch lines through $(0, 0)$ with slopes $\frac{1}{3}, \frac{1}{2}, -\frac{1}{3},$ and 3 .

15–18 ■ Find an equation for the line whose graph is sketched.

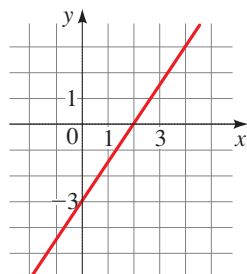
15.



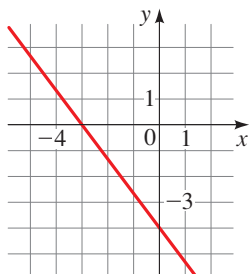
16.



17.



18.



19–38 ■ Find an equation of the line that satisfies the given conditions.

19. Through $(2, 3)$; slope 5
 20. Through $(-2, 4)$; slope -1
 21. Through $(1, 7)$; slope $\frac{2}{3}$
 22. Through $(-3, -5)$; slope $-\frac{7}{2}$
 23. Through $(2, 1)$ and $(1, 6)$
 24. Through $(-1, -2)$ and $(4, 3)$
 25. Slope 3 ; y -intercept -2
 26. Slope $\frac{2}{5}$; y -intercept 4
 27. x -intercept 1 ; y -intercept -3
 28. x -intercept -8 ; y -intercept 6

29. Through $(4, 5)$; parallel to the x -axis
 30. Through $(4, 5)$; parallel to the y -axis
 31. Through $(1, -6)$; parallel to the line $x + 2y = 6$
 32. y -intercept 6 ; parallel to the line $2x + 3y + 4 = 0$
 33. Through $(-1, 2)$; parallel to the line $x = 5$
 34. Through $(2, 6)$; perpendicular to the line $y = 1$
 35. Through $(-1, -2)$; perpendicular to the line $2x + 5y + 8 = 0$
 36. Through $(\frac{1}{2}, -\frac{2}{3})$; perpendicular to the line $4x - 8y = 1$
 37. Through $(1, 7)$; parallel to the line passing through $(2, 5)$ and $(-2, 1)$
 38. Through $(-2, -11)$; perpendicular to the line passing through $(1, 1)$ and $(5, -1)$
 39. (a) Sketch the line with slope $\frac{3}{2}$ that passes through the point $(-2, 1)$.
 (b) Find an equation for this line.
 40. (a) Sketch the line with slope -2 that passes through the point $(4, -1)$.
 (b) Find an equation for this line.



41–44 ■ Use a graphing device to graph the given family of lines in the same viewing rectangle. What do the lines have in common?

41. $y = -2x + b$ for $b = 0, \pm 1, \pm 3, \pm 6$
 42. $y = mx - 3$ for $m = 0, \pm 0.25, \pm 0.75, \pm 1.5$
 43. $y = m(x - 3)$ for $m = 0, \pm 0.25, \pm 0.75, \pm 1.5$
 44. $y = 2 + m(x + 3)$ for $m = 0, \pm 0.5, \pm 1, \pm 2, \pm 6$

45–56 ■ Find the slope and y -intercept of the line and draw its graph.

45. $x + y = 3$
 46. $3x - 2y = 12$
 47. $x + 3y = 0$
 48. $2x - 5y = 0$
 49. $\frac{1}{2}x - \frac{1}{3}y + 1 = 0$
 50. $-3x - 5y + 30 = 0$
 51. $y = 4$
 52. $x = -5$
 53. $3x - 4y = 12$
 54. $4y + 8 = 0$
 55. $3x + 4y - 1 = 0$
 56. $4x + 5y = 10$
 57. Use slopes to show that $A(1, 1)$, $B(7, 4)$, $C(5, 10)$, and $D(-1, 7)$ are vertices of a parallelogram.
 58. Use slopes to show that $A(-3, -1)$, $B(3, 3)$, and $C(-9, 8)$ are vertices of a right triangle.
 59. Use slopes to show that $A(1, 1)$, $B(11, 3)$, $C(10, 8)$, and $D(0, 6)$ are vertices of a rectangle.
 60. Use slopes to determine whether the given points are collinear (lie on a line).
 (a) $(1, 1)$, $(3, 9)$, $(6, 21)$
 (b) $(-1, 3)$, $(1, 7)$, $(4, 15)$
 61. Find an equation of the perpendicular bisector of the line segment joining the points $A(1, 4)$ and $B(7, -2)$.

62. Find the area of the triangle formed by the coordinate axes and the line

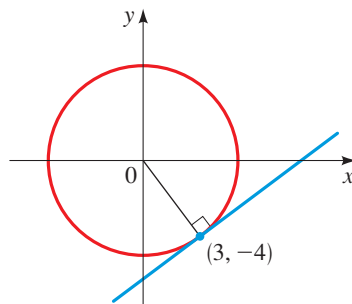
$$2y + 3x - 6 = 0$$

63. (a) Show that if the x - and y -intercepts of a line are nonzero numbers a and b , then the equation of the line can be written in the form

$$\frac{x}{a} + \frac{y}{b} = 1$$

This is called the **two-intercept form** of the equation of a line.

- (b) Use part (a) to find an equation of the line whose x -intercept is 6 and whose y -intercept is -8 .
64. (a) Find an equation for the line tangent to the circle $x^2 + y^2 = 25$ at the point $(3, -4)$. (See the figure.)
- (b) At what other point on the circle will a tangent line be parallel to the tangent line in part (a)?



APPLICATIONS

65. **Grade of a Road** West of Albuquerque, New Mexico, Route 40 eastbound is straight and makes a steep descent toward the city. The highway has a 6% grade, which means that its slope is $-\frac{6}{100}$. Driving on this road, you notice from elevation signs that you have descended a distance of 1000 ft. What is the change in your horizontal distance?



66. **Global Warming** Some scientists believe that the average surface temperature of the world has been rising steadily. The average surface temperature can be modeled by

$$T = 0.02t + 15.0$$

where T is temperature in $^{\circ}\text{C}$ and t is years since 1950.

- (a) What do the slope and T -intercept represent?
- (b) Use the equation to predict the average global surface temperature in 2050.

67. **Drug Dosages** If the recommended adult dosage for a drug is D (in mg), then to determine the appropriate dosage c for a child of age a , pharmacists use the equation

$$c = 0.0417D(a + 1)$$

Suppose the dosage for an adult is 200 mg.

- (a) Find the slope. What does it represent?
- (b) What is the dosage for a newborn?

68. **Flea Market** The manager of a weekend flea market knows from past experience that if she charges x dollars for a rental space at the flea market, then the number y of spaces she can rent is given by the equation $y = 200 - 4x$.

- (a) Sketch a graph of this linear equation. (Remember that the rental charge per space and the number of spaces rented must both be nonnegative quantities.)
- (b) What do the slope, the y -intercept, and the x -intercept of the graph represent?

69. **Production Cost** A small-appliance manufacturer finds that if he produces x toaster ovens in a month, his production cost is given by the equation

$$y = 6x + 3000$$

(where y is measured in dollars).

- (a) Sketch a graph of this linear equation.
- (b) What do the slope and y -intercept of the graph represent?

70. **Temperature Scales** The relationship between the Fahrenheit (F) and Celsius (C) temperature scales is given by the equation $F = \frac{9}{5}C + 32$.

- (a) Complete the table to compare the two scales at the given values.
- (b) Find the temperature at which the scales agree. [Hint: Suppose that a is the temperature at which the scales agree. Set $F = a$ and $C = a$. Then solve for a .]

C	F
-30°	
-20°	
-10°	
0°	
	50°
	68°
	86°

71. **Crickets and Temperature** Biologists have observed that the chirping rate of crickets of a certain species is related to temperature, and the relationship appears to be very nearly linear. A cricket produces 120 chirps per minute at 70°F and 168 chirps per minute at 80°F .

- (a) Find the linear equation that relates the temperature t and the number of chirps per minute n .
- (b) If the crickets are chirping at 150 chirps per minute, estimate the temperature.

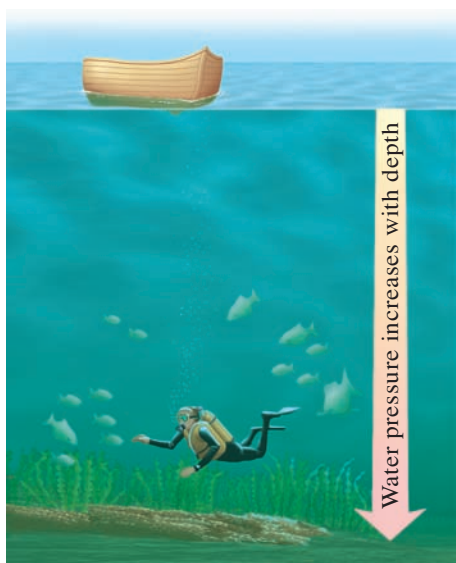
72. **Depreciation** A small business buys a computer for \$4000. After 4 years the value of the computer is expected to be \$200. For accounting purposes the business uses *linear depreciation* to assess the value of the computer at a given time.

This means that if V is the value of the computer at time t , then a linear equation is used to relate V and t .

- Find a linear equation that relates V and t .
- Sketch a graph of this linear equation.
- What do the slope and V -intercept of the graph represent?
- Find the depreciated value of the computer 3 years from the date of purchase.

 **73. Pressure and Depth** At the surface of the ocean the water pressure is the same as the air pressure above the water, 15 lb/in². Below the surface the water pressure increases by 4.34 lb/in² for every 10 ft of descent.

- Find an equation for the relationship between pressure and depth below the ocean surface.
- Sketch a graph of this linear equation.
- What do the slope and y -intercept of the graph represent?
- At what depth is the pressure 100 lb/in²?



74. Distance, Speed, and Time Jason and Debbie leave Detroit at 2:00 P.M. and drive at a constant speed, traveling

west on I-90. They pass Ann Arbor, 40 mi from Detroit, at 2:50 P.M.

- Express the distance traveled in terms of the time elapsed.
- Draw the graph of the equation in part (a).
- What is the slope of this line? What does it represent?

75. Cost of Driving The monthly cost of driving a car depends on the number of miles driven. Lynn found that in May her driving cost was \$380 for 480 mi and in June her cost was \$460 for 800 mi. Assume that there is a linear relationship between the monthly cost C of driving a car and the distance driven d .

- Find a linear equation that relates C and d .
- Use part (a) to predict the cost of driving 1500 mi per month.
- Draw the graph of the linear equation. What does the slope of the line represent?
- What does the y -intercept of the graph represent?
- Why is a linear relationship a suitable model for this situation?

76. Manufacturing Cost The manager of a furniture factory finds that it costs \$2200 to manufacture 100 chairs in one day and \$4800 to produce 300 chairs in one day.

- Assuming that the relationship between cost and the number of chairs produced is linear, find an equation that expresses this relationship. Then graph the equation.
- What is the slope of the line in part (a), and what does it represent?
- What is the y -intercept of this line, and what does it represent?

DISCOVERY ■ DISCUSSION ■ WRITING

77. What Does the Slope Mean? Suppose that the graph of the outdoor temperature over a certain period of time is a line. How is the weather changing if the slope of the line is positive? If it is negative? If it is zero?

78. Collinear Points Suppose that you are given the coordinates of three points in the plane and you want to see whether they lie on the same line. How can you do this using slopes? Using the Distance Formula? Can you think of another method?

1.11 MAKING MODELS USING VARIATION

| Direct Variation ► Inverse Variation ► Joint Variation

When scientists talk about a mathematical model for a real-world phenomenon, they often mean an equation that describes the relationship between two quantities. For instance, the model might describe how the population of an animal species varies with time or how the pressure of a gas varies as its temperature changes. In this section we study a kind of modeling called *variation*.

▼ Direct Variation

Two types of mathematical models occur so often that they are given special names. The first is called *direct variation* and occurs when one quantity is a constant multiple of the other, so we use an equation of the form $y = kx$ to model this dependence.

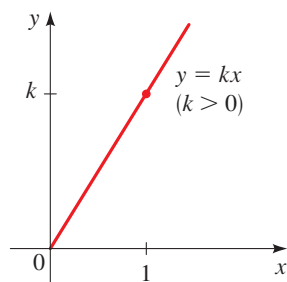


FIGURE 1

DIRECT VARIATION

If the quantities x and y are related by an equation

$$y = kx$$

for some constant $k \neq 0$, we say that y **varies directly as x** , or y is **directly proportional to x** , or simply y is **proportional to x** . The constant k is called the **constant of proportionality**.

Recall that the graph of an equation of the form $y = mx + b$ is a line with slope m and y -intercept b . So the graph of an equation $y = kx$ that describes direct variation is a line with slope k and y -intercept 0 (see Figure 1).

EXAMPLE 1 | Direct Variation



During a thunderstorm you see the lightning before you hear the thunder because light travels much faster than sound. The distance between you and the storm varies directly as the time interval between the lightning and the thunder.

- Suppose that the thunder from a storm 5400 ft away takes 5 s to reach you. Determine the constant of proportionality, and write the equation for the variation.
- Sketch the graph of this equation. What does the constant of proportionality represent?
- If the time interval between the lightning and thunder is now 8 s, how far away is the storm?

SOLUTION

- Let d be the distance from you to the storm, and let t be the length of the time interval. We are given that d varies directly as t , so

$$d = kt$$

where k is a constant. To find k , we use the fact that $t = 5$ when $d = 5400$. Substituting these values in the equation, we get

$$5400 = k(5) \quad \text{Substitute}$$

$$k = \frac{5400}{5} = 1080 \quad \text{Solve for } k$$

Substituting this value of k in the equation for d , we obtain

$$d = 1080t$$

as the equation for d as a function of t .

- The graph of the equation $d = 1080t$ is a line through the origin with slope 1080 and is shown in Figure 2. The constant $k = 1080$ is the approximate speed of sound (in ft/s).
- When $t = 8$, we have

$$d = 1080 \cdot 8 = 8640$$

So the storm is 8640 ft $\approx$ 1.6 mi away.

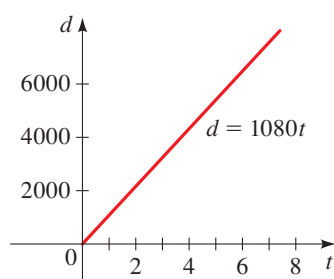


FIGURE 2

 **NOW TRY EXERCISES 17 AND 29**

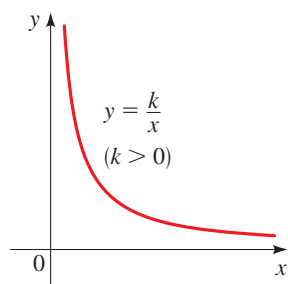


FIGURE 3 Inverse variation

▼ Inverse Variation

Another equation that is frequently used in mathematical modeling is $y = k/x$, where k is a constant.

INVERSE VARIATION

If the quantities x and y are related by the equation

$$y = \frac{k}{x}$$

for some constant $k \neq 0$, we say that y is **inversely proportional to x** or y **varies inversely as x** . The constant k is called the **constant of proportionality**.

The graph of $y = k/x$ for $x > 0$ is shown in Figure 3 for the case $k > 0$. It gives a picture of what happens when y is inversely proportional to x .

EXAMPLE 2 | Inverse Variation

Boyle's Law states that when a sample of gas is compressed at a constant temperature, the pressure of the gas is inversely proportional to the volume of the gas.

- Suppose the pressure of a sample of air that occupies 0.106 m^3 at 25°C is 50 kPa . Find the constant of proportionality, and write the equation that expresses the inverse proportionality.
- If the sample expands to a volume of 0.3 m^3 , find the new pressure.

SOLUTION

- Let P be the pressure of the sample of gas, and let V be its volume. Then, by the definition of inverse proportionality, we have

$$P = \frac{k}{V}$$

where k is a constant. To find k , we use the fact that $P = 50$ when $V = 0.106$. Substituting these values in the equation, we get

$$50 = \frac{k}{0.106} \quad \text{Substitute}$$

$$k = (50)(0.106) = 5.3 \quad \text{Solve for } k$$

Putting this value of k in the equation for P , we have

$$P = \frac{5.3}{V}$$

- When $V = 0.3$, we have

$$P = \frac{5.3}{0.3} \approx 17.7$$

So the new pressure is about 17.7 kPa .

 **NOW TRY EXERCISES 19 AND 35**

▼ Joint Variation

A physical quantity often depends on more than one other quantity. If one quantity is proportional to two or more other quantities, we call this relationship *joint variation*.

JOINT VARIATION

If the quantities x , y , and z are related by the equation

$$z = kxy$$

where k is a nonzero constant, we say that z **varies jointly as x and y** or z is **jointly proportional to x and y** .

In the sciences, relationships between three or more variables are common, and any combination of the different types of proportionality that we have discussed is possible. For example, if

$$z = k \frac{x}{y}$$

we say that z is **proportional to x** and **inversely proportional to y** .

EXAMPLE 3 | Newton's Law of Gravitation

Newton's Law of Gravitation says that two objects with masses m_1 and m_2 attract each other with a force F that is jointly proportional to their masses and inversely proportional to the square of the distance r between the objects. Express Newton's Law of Gravitation as an equation.

SOLUTION Using the definitions of joint and inverse variation and the traditional notation G for the gravitational constant of proportionality, we have

$$F = G \frac{m_1 m_2}{r^2}$$

 **NOW TRY EXERCISES 21 AND 41**

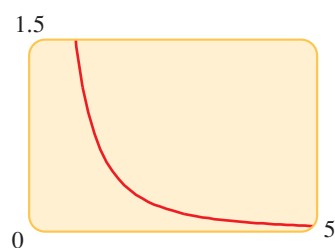


FIGURE 4 Graph of $F = \frac{1}{r^2}$

If m_1 and m_2 are fixed masses, then the gravitational force between them is $F = C/r^2$ (where $C = Gm_1m_2$ is a constant). Figure 4 shows the graph of this equation for $r > 0$ with $C = 1$. Observe how the gravitational attraction decreases with increasing distance.

1.11 EXERCISES**CONCEPTS**

- If the quantities x and y are related by the equation $y = 3x$, then we say that y is _____ to x and the constant of _____ is 3.
- If the quantities x and y are related by the equation $y = \frac{3}{x}$, then we say that y is _____ to x and the constant of _____ is 3.
- If the quantities x , y , and z are related by the equation $z = 3 \frac{x}{y}$, then we say that z is _____ to x and _____ to y .

- If z is jointly proportional to x and y and if z is 10 when x is 4 and y is 5, then x , y , and z are related by the equation $z = \underline{\hspace{2cm}}$.

SKILLS

5–16 ■ Write an equation that expresses the statement.

- T varies directly as x .
- P is directly proportional to w .
- v is inversely proportional to z .
- w is jointly proportional to m and n .
- y is proportional to s and inversely proportional to t .

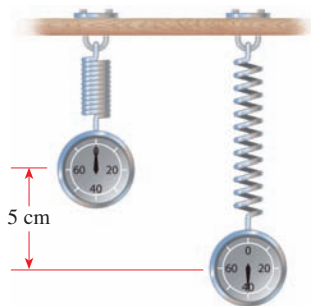
10. P varies inversely as T .
11. z is proportional to the square root of y .
12. A is proportional to the square of t and inversely proportional to the cube of x .
13. V is jointly proportional to l , w , and h .
14. S is jointly proportional to the squares of r and θ .
15. R is proportional to i and inversely proportional to P and t .
16. A is jointly proportional to the square roots of x and y .

17–28 ■ Express the statement as an equation. Use the given information to find the constant of proportionality.

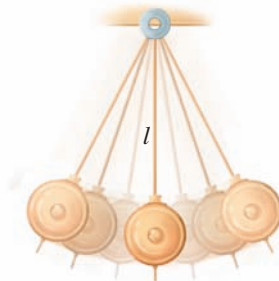
17. y is directly proportional to x . If $x = 6$, then $y = 42$.
18. z varies inversely as t . If $t = 3$, then $z = 5$.
19. R is inversely proportional to s . If $s = 4$, then $R = 3$.
20. P is directly proportional to T . If $T = 300$, then $P = 20$.
21. M varies directly as x and inversely as y . If $x = 2$ and $y = 6$, then $M = 5$.
22. S varies jointly as p and q . If $p = 4$ and $q = 5$, then $S = 180$.
23. W is inversely proportional to the square of r . If $r = 6$, then $W = 10$.
24. t is jointly proportional to x and y , and inversely proportional to r . If $x = 2$, $y = 3$, and $r = 12$, then $t = 25$.
25. C is jointly proportional to l , w , and h . If $l = w = h = 2$, then $C = 128$.
26. H is jointly proportional to the squares of l and w . If $l = 2$ and $w = \frac{1}{3}$, then $H = 36$.
27. s is inversely proportional to the square root of t . If $s = 100$, then $t = 25$.
28. M is jointly proportional to a , b , and c and inversely proportional to d . If a and d have the same value and if b and c are both 2, then $M = 128$.

APPLICATIONS

29. **Hooke's Law** Hooke's Law states that the force needed to keep a spring stretched x units beyond its natural length is directly proportional to x . Here the constant of proportionality is called the **spring constant**.
 - (a) Write Hooke's Law as an equation.
 - (b) If a spring has a natural length of 10 cm and a force of 40 N is required to maintain the spring stretched to a length of 15 cm, find the spring constant.
 - (c) What force is needed to keep the spring stretched to a length of 14 cm?



30. **Law of the Pendulum** The period of a pendulum (the time elapsed during one complete swing of the pendulum) varies directly with the square root of the length of the pendulum.
 - (a) Express this relationship by writing an equation.
 - (b) To double the period, how would we have to change the length l ?



31. **Printing Costs** The cost C of printing a magazine is jointly proportional to the number of pages p in the magazine and the number of magazines printed m .
 - (a) Write an equation that expresses this joint variation.
 - (b) Find the constant of proportionality if the printing cost is \$60,000 for 4000 copies of a 120-page magazine.
 - (c) How much would the printing cost be for 5000 copies of a 92-page magazine?
32. **Boyle's Law** The pressure P of a sample of gas is directly proportional to the temperature T and inversely proportional to the volume V .
 - (a) Write an equation that expresses this variation.
 - (b) Find the constant of proportionality if 100 L of gas exerts a pressure of 33.2 kPa at a temperature of 400 K (absolute temperature measured on the Kelvin scale).
 - (c) If the temperature is increased to 500 K and the volume is decreased to 80 L, what is the pressure of the gas?
33. **Power from a Windmill** The power P that can be obtained from a windmill is directly proportional to the cube of the wind speed s .
 - (a) Write an equation that expresses this variation.
 - (b) Find the constant of proportionality for a windmill that produces 96 watts of power when the wind is blowing at 20 mi/h.
 - (c) How much power will this windmill produce if the wind speed increases to 30 mi/h?
34. **Power Needed to Propel a Boat** The power P (measured in horse power, hp) needed to propel a boat is directly proportional to the cube of the speed s . An 80-hp engine is needed to propel a certain boat at 10 knots. Find the power needed to drive the boat at 15 knots.



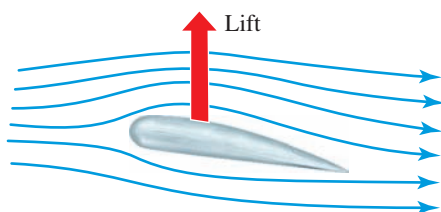
-  **35. Loudness of Sound** The loudness L of a sound (measured in decibels, dB) is inversely proportional to the square of the distance d from the source of the sound. A person who is 10 ft from a lawn mower experiences a sound level of 70 dB. How loud is the lawn mower when the person is 100 ft away?

- 36. Stopping Distance** The stopping distance D of a car after the brakes have been applied varies directly as the square of the speed s . A certain car traveling at 50 mi/h can stop in 240 ft. What is the maximum speed it can be traveling if it needs to stop in 160 ft?

- 37. A Jet of Water** The power P of a jet of water is jointly proportional to the cross-sectional area A of the jet and to the cube of the velocity v . If the velocity is doubled and the cross-sectional area is halved, by what factor will the power increase?



- 38. Aerodynamic Lift** The lift L on an airplane wing at take-off varies jointly as the square of the speed s of the plane and the area A of its wings. A plane with a wing area of 500 ft² traveling at 50 mi/h experiences a lift of 1700 lb. How much lift would a plane with a wing area of 600 ft² traveling at 40 mi/h experience?



- 39. Drag Force on a Boat** The drag force F on a boat is jointly proportional to the wetted surface area A on the hull and the square of the speed s of the boat. A boat experiences a drag force of 220 lb when traveling at 5 mi/h with a wetted surface area of 40 ft². How fast must a boat be traveling if it has 28 ft² of wetted surface area and is experiencing a drag force of 175 lb?

- 40. Skidding in a Curve** A car is traveling on a curve that forms a circular arc. The force F needed to keep the car from skidding is jointly proportional to the weight w of the car and the square of its speed s , and is inversely proportional to the radius r of the curve.

(a) Write an equation that expresses this variation.

- (b) A car weighing 1600 lb travels around a curve at 60 mi/h. The next car to round this curve weighs 2500 lb and requires the same force as the first car to keep from skidding. How fast is the second car traveling?



-  **41. Electrical Resistance** The resistance R of a wire varies directly as its length L and inversely as the square of its diameter d .

- (a) Write an equation that expresses this joint variation.
 (b) Find the constant of proportionality if a wire 1.2 m long and 0.005 m in diameter has a resistance of 140 ohms.
 (c) Find the resistance of a wire made of the same material that is 3 m long and has a diameter of 0.008 m.

- 42. Kepler's Third Law** Kepler's Third Law of planetary motion states that the square of the period T of a planet (the time it takes for the planet to make a complete revolution about the sun) is directly proportional to the cube of its average distance d from the sun.

- (a) Express Kepler's Third Law as an equation.
 (b) Find the constant of proportionality by using the fact that for our planet the period is about 365 days and the average distance is about 93 million miles.
 (c) The planet Neptune is about 2.79×10^9 mi from the sun. Find the period of Neptune.

- 43. Radiation Energy** The total radiation energy E emitted by a heated surface per unit area varies as the fourth power of its absolute temperature T . The temperature is 6000 K at the surface of the sun and 300 K at the surface of the earth.

- (a) How many times more radiation energy per unit area is produced by the sun than by the earth?
 (b) The radius of the earth is 3960 mi and the radius of the sun is 435,000 mi. How many times more total radiation does the sun emit than the earth?

- 44. Value of a Lot** The value of a building lot on Galiano Island is jointly proportional to its area and the quantity of water produced by a well on the property. A 200 ft by 300 ft lot has a well producing 10 gallons of water per minute, and is valued at \$48,000. What is the value of a 400 ft by 400 ft lot if the well on the lot produces 4 gallons of water per minute?

- 45. Growing Cabbages** In the short growing season of the Canadian arctic territory of Nunavut, some gardeners find it possible to grow gigantic cabbages in the midnight sun.

Assume that the final size of a cabbage is proportional to the amount of nutrients it receives and inversely proportional to the number of other cabbages surrounding it. A cabbage that received 20 oz of nutrients and had 12 other cabbages around it grew to 30 lb. What size would it grow to if it received 10 oz of nutrients and had only 5 cabbage “neighbors”?

- 46. Heat of a Campfire** The heat experienced by a hiker at a campfire is proportional to the amount of wood on the fire and inversely proportional to the cube of his distance from the fire. If the hiker is 20 ft from the fire and someone doubles the amount of wood burning, how far from the fire would he have to be so that he feels the same heat as before?



- 47. Frequency of Vibration** The frequency f of vibration of a violin string is inversely proportional to its length L . The constant of proportionality k is positive and depends on the tension and density of the string.

- Write an equation that represents this variation.
- What effect does doubling the length of the string have on the frequency of its vibration?

- 48. Spread of a Disease** The rate r at which a disease spreads in a population of size P is jointly proportional to the number x of infected people and the number $P - x$ who are not infected. An infection erupts in a small town that has population $P = 5000$.

- Write an equation that expresses r as a function of x .
- Compare the rate of spread of this infection when 10 people are infected to the rate of spread when 1000 people are infected. Which rate is larger? By what factor?
- Calculate the rate of spread when the entire population is infected. Why does this answer make intuitive sense?

DISCOVERY ■ DISCUSSION ■ WRITING

- 49. Is Proportionality Everything?** A great many laws of physics and chemistry are expressible as proportionalities. Give at least one example of a function that occurs in the sciences that is *not* a proportionality.

CHAPTER 1 | REVIEW

■ CONCEPT CHECK

- Define each term in your own words. (Check by referring to the definition in the text.)
 - An integer
 - A rational number
 - An irrational number
 - A real number
- State each of these properties of real numbers.
 - Commutative Property
 - Associative Property
 - Distributive Property
- What is an open interval? What is a closed interval? What notation is used for these intervals?
- What is the absolute value of a number?
- In the expression a^x , which is the base and which is the exponent?
 - What does a^x mean if $x = n$, a positive integer?
 - What if $x = 0$?
 - What if x is a negative integer: $x = -n$, where n is a positive integer?
 - What if $x = m/n$, a rational number?
 - State the Laws of Exponents.
- What does $\sqrt[n]{a} = b$ mean?
 - Why is $\sqrt{a^2} = |a|$?
 - How many real n th roots does a positive real number have if n is odd? If n is even?
- Explain how the procedure of rationalizing the denominator works.
- State the Special Product Formulas for $(a + b)^2$, $(a - b)^2$, $(a + b)^3$, and $(a - b)^3$.
- State each Special Factoring Formula.
 - Difference of squares
 - Difference of cubes
 - Sum of cubes
- What is a solution of an equation?
- How do you solve an equation involving radicals? Why is it important to check your answers when solving equations of this type?
- How do you solve an equation
 - algebraically?
 - graphically?
- Write the general form of each type of equation.
 - A linear equation
 - A quadratic equation
- What are the three ways to solve a quadratic equation?
- State the Zero-Product Property.
- Describe the process of completing the square.
- State the quadratic formula.
- What is the discriminant of a quadratic equation?
- State the rules for working with inequalities.

20. How do you solve
 (a) a linear inequality?
 (b) a nonlinear inequality?
21. (a) How do you solve an equation involving an absolute value?
 (b) How do you solve an inequality involving an absolute value?
22. (a) Describe the coordinate plane.
 (b) How do you locate points in the coordinate plane?
23. State each formula.
 (a) The Distance Formula
 (b) The Midpoint Formula
24. Given an equation, what is its graph?
25. How do you find the x -intercepts and y -intercepts of a graph?
26. Write an equation of the circle with center (h, k) and radius r .
27. Explain the meaning of each type of symmetry. How do you test for it?
 (a) Symmetry with respect to the x -axis
 (b) Symmetry with respect to the y -axis
 (c) Symmetry with respect to the origin
28. Define the slope of a line.
29. Write each form of the equation of a line.
 (a) The point-slope form
 (b) The slope-intercept form
30. (a) What is the equation of a vertical line?
 (b) What is the equation of a horizontal line?
31. What is the general equation of a line?
32. Given lines with slopes m_1 and m_2 , explain how you can tell if the lines are
 (a) parallel (b) perpendicular
33. Write an equation that expresses each relationship.
 (a) y is directly proportional to x .
 (b) y is inversely proportional to x .
 (c) z is jointly proportional to x and y .

EXERCISES

1–4 ■ State the property of real numbers being used.

1. $3x + 2y = 2y + 3x$
2. $(a + b)(a - b) = (a - b)(a + b)$
3. $4(a + b) = 4a + 4b$
4. $(A + 1)(x + y) = (A + 1)x + (A + 1)y$

5–6 ■ Express the interval in terms of inequalities, and then graph the interval.

5. $[-2, 6]$
6. $(-\infty, 4]$

7–8 ■ Express the inequality in interval notation, and then graph the corresponding interval.

7. $x \geq 5$
8. $-1 < x \leq 5$

9–18 ■ Evaluate the expression.

9. $|3 - |-9||$
10. $1 - |1 - |-1||$
11. $2^{-3} - 3^{-2}$
12. $\sqrt[3]{-125}$
13. $216^{-1/3}$
14. $64^{2/3}$
15. $\frac{\sqrt{242}}{\sqrt{2}}$
16. $\sqrt[4]{4} \sqrt[4]{324}$
17. $2^{1/2} 8^{1/2}$
18. $\sqrt{2} \sqrt{50}$

19–28 ■ Simplify the expression.

19. $\frac{x^2(2x)^4}{x^3}$
20. $(a^2)^{-3}(a^3b)^2(b^3)^4$
21. $(3xy^2)^3(\frac{2}{3}x^{-1}y)^2$
22. $\left(\frac{r^2s^{4/3}}{r^{1/3}s}\right)^6$
23. $\sqrt[3]{(x^3y^2)^4}$
24. $\sqrt{x^2y^4}$
25. $\left(\frac{9x^3y}{y^{-3}}\right)^{1/2}$
26. $\left(\frac{x^{-2}y^3}{x^2y}\right)^{-1/2} \left(\frac{x^3y}{y^{1/2}}\right)^2$

$$27. \frac{8r^{1/2}s^{-3}}{2r^{-2}s^4} \quad 28. \left(\frac{ab^2c^{-3}}{2a^3b^{-4}}\right)^{-2}$$

29. Write the number 78,250,000,000 in scientific notation.

30. Write the number 2.08×10^{-8} in ordinary decimal notation.

31. If $a \approx 0.00000293$, $b \approx 1.582 \times 10^{-14}$, and $c \approx 2.8064 \times 10^{12}$, use a calculator to approximate the number ab/c .

32. If your heart beats 80 times per minute and you live to be 90 years old, estimate the number of times your heart beats during your lifetime. State your answer in scientific notation.

33–48 ■ Factor the expression completely.

33. $12x^2y^4 - 3xy^5 + 9x^3y^2$
34. $x^2 - 9x + 18$
35. $x^2 + 3x - 10$
36. $6x^2 + x - 12$
37. $4t^2 - 13t - 12$
38. $x^4 - 2x^2 + 1$
39. $25 - 16t^2$
40. $2y^6 - 32y^2$
41. $x^6 - 1$
42. $y^3 - 2y^2 - y + 2$
43. $x^{-1/2} - 2x^{1/2} + x^{3/2}$
44. $a^4b^2 + ab^5$
45. $4x^3 - 8x^2 + 3x - 6$
46. $8x^3 + y^6$
47. $(x^2 + 2)^{5/2} + 2x(x^2 + 2)^{3/2} + x^2\sqrt{x^2 + 2}$
48. $3x^3 - 2x^2 + 18x - 12$

49–64 ■ Perform the indicated operations and simplify.

49. $(2x + 1)(3x - 2) - 5(4x - 1)$
50. $(2y - 7)(2y + 7)$
51. $(1 + x)(2 - x) - (3 - x)(3 + x)$
52. $\sqrt{x}(\sqrt{x} + 1)(2\sqrt{x} - 1)$
53. $x^2(x - 2) + x(x - 2)^2$
54. $\frac{x^2 - 2x - 3}{2x^2 + 5x + 3}$

$$55. \frac{x^2 + 2x - 3}{x^2 + 8x + 16} \cdot \frac{3x + 12}{x - 1} \quad 56. \frac{t^3 - 1}{t^2 - 1}$$

$$57. \frac{x^2 - 2x - 15}{x^2 - 6x + 5} \div \frac{x^2 - x - 12}{x^2 - 1}$$

$$58. \frac{2}{x} + \frac{1}{x-2} + \frac{3}{(x-2)^2} \quad 59. \frac{1}{x-1} - \frac{2}{x^2-1}$$

$$60. \frac{1}{x+2} + \frac{1}{x^2-4} - \frac{2}{x^2-x-2}$$

$$61. \frac{\frac{1}{x} - \frac{1}{2}}{x-2} \quad 62. \frac{\frac{1}{x} - \frac{1}{x+1}}{\frac{1}{x} + \frac{1}{x+1}}$$

$$63. \frac{\sqrt{6}}{\sqrt{3} + \sqrt{2}} \quad (\text{rationalize the denominator})$$

$$64. \frac{\sqrt{x+h} - \sqrt{x}}{h} \quad (\text{rationalize the numerator})$$

65–80 ■ Find all real solutions of the equation.

$$65. 7x - 6 = 4x + 9$$

$$66. 8 - 2x = 14 + x$$

$$67. \frac{x+1}{x-1} = \frac{3x}{3x-6}$$

$$68. (x+2)^2 = (x-4)^2$$

$$69. x^2 - 9x + 14 = 0$$

$$70. x^2 + 24x + 144 = 0$$

$$71. 2x^2 + x = 1$$

$$72. 3x^2 + 5x - 2 = 0$$

$$73. 4x^3 - 25x = 0$$

$$74. x^3 - 2x^2 - 5x + 10 = 0$$

$$75. 3x^2 + 4x - 1 = 0$$

$$76. \frac{1}{x} + \frac{2}{x-1} = 3$$

$$77. \frac{x}{x-2} + \frac{1}{x+2} = \frac{8}{x^2-4}$$

$$78. x^4 - 8x^2 - 9 = 0$$

$$79. |x-7| = 4$$

$$80. |2x-5| = 9$$

81. The owner of a store sells raisins for \$3.20 per pound and nuts for \$2.40 per pound. He decides to mix the raisins and nuts and sell 50 lb of the mixture for \$2.72 per pound. What quantities of raisins and nuts should he use?

82. Anthony leaves Kingstown at 2:00 P.M. and drives to Queensville, 160 mi distant, at 45 mi/h. At 2:15 P.M. Helen leaves Queensville and drives to Kingstown at 40 mi/h. At what time do they pass each other on the road?

83. A woman cycles 8 mi/h faster than she runs. Every morning she cycles 4 mi and runs $2\frac{1}{2}$ mi, for a total of one hour of exercise. How fast does she run?

84. The hypotenuse of a right triangle has length 20 cm. The sum of the lengths of the other two sides is 28 cm. Find the lengths of the other two sides of the triangle.

85. Abbie paints twice as fast as Beth and three times as fast as Cathie. If it takes them 60 min to paint a living room with all three working together, how long would it take Abbie if she works alone?

86. A homeowner wishes to fence in three adjoining garden plots, one for each of her children, as shown in the figure. If each

plot is to be 80 ft² in area and she has 88 ft of fencing material at hand, what dimensions should each plot have?



87–94 ■ Solve the inequality. Express the solution using interval notation and graph the solution set on the real number line.

$$87. 3x - 2 > -11$$

$$88. -1 < 2x + 5 \leq 3$$

$$89. x^2 + 4x - 12 > 0$$

$$90. x^2 \leq 1$$

$$91. \frac{x-4}{x^2-4} \leq 0$$

$$92. \frac{5}{x^3 - x^2 - 4x + 4} < 0$$

$$93. |x-5| \leq 3$$

$$94. |x-4| < 0.02$$



95–98 ■ Solve the equation or inequality graphically.

$$95. x^2 - 4x = 2x + 7$$

$$96. \sqrt{x+4} = x^2 - 5$$

$$97. 4x - 3 \geq x^2$$

$$98. x^3 - 4x^2 - 5x > 2$$

99–100 ■ Two points P and Q are given.

(a) Plot P and Q on a coordinate plane.

(b) Find the distance from P to Q .

(c) Find the midpoint of the segment PQ .

(d) Sketch the line determined by P and Q , and find its equation in slope-intercept form.

(e) Sketch the circle that passes through Q and has center P , and find the equation of this circle.

$$99. P(2, 0), \quad Q(-5, 12)$$

$$100. P(7, -1), \quad Q(2, -11)$$

101–102 ■ Sketch the region given by the set.

$$101. \{(x, y) \mid -4 < x < 4 \text{ and } -2 < y < 2\}$$

$$102. \{(x, y) \mid x \geq 4 \text{ or } y \geq 2\}$$

103. Which of the points $A(4, 4)$ or $B(5, 3)$ is closer to the point $C(-1, -3)$?

104. Find an equation of the circle that has center $(2, -5)$ and radius $\sqrt{2}$.

105. Find an equation of the circle that has center $(-5, -1)$ and passes through the origin.

106. Find an equation of the circle that contains the points $P(2, 3)$ and $Q(-1, 8)$ and has the midpoint of the segment PQ as its center.

107–110 ■ Determine whether the equation represents a circle, represents a point, or has no graph. If the equation is that of a circle, find its center and radius.

$$107. x^2 + y^2 + 2x - 6y + 9 = 0$$

$$108. 2x^2 + 2y^2 - 2x + 8y = \frac{1}{2}$$

109. $x^2 + y^2 + 72 = 12x$

110. $x^2 + y^2 - 6x - 10y + 34 = 0$

111–118 ■ Test the equation for symmetry, and sketch its graph.

111. $y = 2 - 3x$

112. $2x - y + 1 = 0$

113. $x + 3y = 21$

114. $x = 2y + 12$

115. $y = 16 - x^2$

116. $8x + y^2 = 0$

117. $x = \sqrt{y}$

118. $y = -\sqrt{1 - x^2}$



119–122 ■ Use a graphing device to graph the equation in an appropriate viewing rectangle.

119. $y = x^2 - 6x$

120. $y = \sqrt{5 - x}$

121. $y = x^3 - 4x^2 - 5x$

122. $\frac{x^2}{4} + y^2 = 1$

123. Find an equation for the line that passes through the points $(-1, -6)$ and $(2, -4)$.

124. Find an equation for the line that passes through the point $(6, -3)$ and has slope $-\frac{1}{2}$.

125. Find an equation for the line that has x -intercept 4 and y -intercept 12.

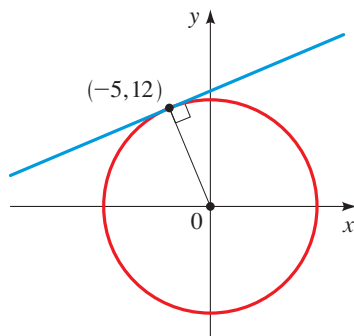
126. Find an equation for the line that passes through the point $(1, 7)$ and is perpendicular to the line $x - 3y + 16 = 0$.

127. Find an equation for the line that passes through the origin and is parallel to the line $3x + 15y = 22$.

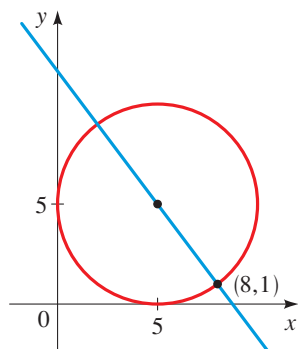
128. Find an equation for the line that passes through the point $(5, 2)$ and is parallel to the line passing through $(-1, -3)$ and $(3, 2)$.

129–130 ■ Find equations for the circle and the line in the figure.

129.



130.



131. Hooke's Law states that if a weight w is attached to a hanging spring, then the stretched length s of the spring is linearly related to w . For a particular spring we have

$$s = 0.3w + 2.5$$

where s is measured in inches and w in pounds.

(a) What do the slope and s -intercept in this equation represent?

(b) How long is the spring when a 5-lb weight is attached?

132. Margarita is hired by an accounting firm at a salary of \$60,000 per year. Three years later her annual salary has increased to \$70,500. Assume that her salary increases linearly.

(a) Find an equation that relates her annual salary S and the number of years t that she has worked for the firm.

(b) What do the slope and S -intercept of her salary equation represent?

(c) What will her salary be after 12 years with the firm?

133. Suppose that M varies directly as z , and $M = 120$ when $z = 15$. Write an equation that expresses this variation.

134. Suppose that z is inversely proportional to y , and that $z = 12$ when $y = 16$. Write an equation that expresses z in terms of y .

135. The intensity of illumination I from a light varies inversely as the square of the distance d from the light.

(a) Write this statement as an equation.

(b) Determine the constant of proportionality if it is known that a lamp has an intensity of 1000 candles at a distance of 8 m.

(c) What is the intensity of this lamp at a distance of 20 m?

136. The frequency of a vibrating string under constant tension is inversely proportional to its length. If a violin string 12 inches long vibrates 440 times per second, to what length must it be shortened to vibrate 660 times per second?

137. The terminal velocity of a parachutist is directly proportional to the square root of his weight. A 160-lb parachutist attains a terminal velocity of 9 mi/h. What is the terminal velocity for a parachutist weighing 240 lb?

138. The maximum range of a projectile is directly proportional to the square of its velocity. A baseball pitcher throws a ball at 60 mi/h, with a maximum range of 242 ft. What is his maximum range if he throws the ball at 70 mi/h?

- (a) Graph the intervals $(-5, 3]$ and $(2, \infty)$ on the real number line.

(b) Express the inequalities $x \leq 3$ and $-1 \leq x < 4$ in interval notation.

(c) Find the distance between -7 and 9 on the real number line.
- Evaluate each expression.

(a) $(-3)^4$ (b) -3^4 (c) 3^{-4} (d) $\frac{5^{23}}{5^{21}}$ (e) $\left(\frac{2}{3}\right)^{-2}$ (f) $16^{-3/4}$
- Write each number in scientific notation.

(a) 186,000,000,000 (b) 0.0000003965
- Simplify each expression. Write your final answer without negative exponents.

(a) $\sqrt{200} - \sqrt{32}$ (b) $(3a^3b^3)(4ab^2)^2$ (c) $\left(\frac{3x^{3/2}y^3}{x^2y^{-1/2}}\right)^{-2}$

(d) $\frac{x^2 + 3x + 2}{x^2 - x - 2}$ (e) $\frac{x^2}{x^2 - 4} - \frac{x + 1}{x + 2}$ (f) $\frac{\frac{y}{x} - \frac{x}{y}}{\frac{1}{y} - \frac{1}{x}}$
- Rationalize the denominator and simplify: $\frac{\sqrt{10}}{\sqrt{5} - 2}$
- Perform the indicated operations and simplify.

(a) $3(x + 6) + 4(2x - 5)$ (b) $(x + 3)(4x - 5)$ (c) $(\sqrt{a} + \sqrt{b})(\sqrt{a} - \sqrt{b})$

(d) $(2x + 3)^2$ (e) $(x + 2)^3$
- Factor each expression completely.

(a) $4x^2 - 25$ (b) $2x^2 + 5x - 12$ (c) $x^3 - 3x^2 - 4x + 12$

(d) $x^4 + 27x$ (e) $3x^{3/2} - 9x^{1/2} + 6x^{-1/2}$ (f) $x^3y - 4xy$
- Find all real solutions.

(a) $x + 5 = 14 - \frac{1}{2}x$ (b) $\frac{2x}{x + 1} = \frac{2x - 1}{x}$ (c) $x^2 - x - 12 = 0$

(d) $2x^2 + 4x + 1 = 0$ (e) $\sqrt{3 - \sqrt{x + 5}} = 2$ (f) $x^4 - 3x^2 + 2 = 0$

(g) $3|x - 4| = 10$
- Mary drove from Amity to Belleville at a speed of 50 mi/h. On the way back, she drove at 60 mi/h. The total trip took $4\frac{2}{3}$ h of driving time. Find the distance between these two cities.
- A rectangular parcel of land is 70 ft longer than it is wide. Each diagonal between opposite corners is 130 ft. What are the dimensions of the parcel?
- Solve each inequality. Write the answer using interval notation, and sketch the solution on the real number line.

(a) $-4 < 5 - 3x \leq 17$ (b) $x(x - 1)(x + 2) > 0$

(c) $|x - 4| < 3$ (d) $\frac{2x - 3}{x + 1} \leq 1$
- A bottle of medicine is to be stored at a temperature between 5°C and 10°C . What range does this correspond to on the Fahrenheit scale? [Note: Fahrenheit (F) and Celsius (C) temperatures satisfy the relation $C = \frac{5}{9}(F - 32)$.]
- For what values of x is the expression $\sqrt{6x - x^2}$ defined as a real number?
-  Solve the equation and the inequality graphically.

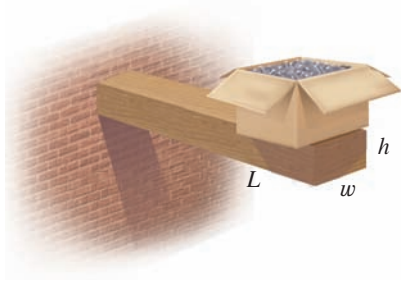
(a) $x^3 - 9x - 1 = 0$ (b) $x^2 - 1 \leq |x + 1|$
- (a) Plot the points $P(0, 3)$, $Q(3, 0)$, and $R(6, 3)$ in the coordinate plane. Where must the point S be located so that $PQRS$ is a square?

(b) Find the area of $PQRS$.
- (a) Sketch the graph of $y = x^2 - 4$.

(b) Find the x - and y -intercepts of the graph.

(c) Is the graph symmetric about the x -axis, the y -axis, or the origin?

17. Let $P(-3, 1)$ and $Q(5, 6)$ be two points in the coordinate plane.
- Plot P and Q in the coordinate plane.
 - Find the distance between P and Q .
 - Find the midpoint of the segment PQ .
 - Find the slope of the line that contains P and Q .
 - Find the perpendicular bisector of the line that contains P and Q .
 - Find an equation for the circle for which the segment PQ is a diameter.
18. Find the center and radius of each circle and sketch its graph.
- $x^2 + y^2 = 25$
 - $(x - 2)^2 + (y + 1)^2 = 9$
 - $x^2 + 6x + y^2 - 2y + 6 = 0$
19. Write the linear equation $2x - 3y = 15$ in slope-intercept form, and sketch its graph. What are the slope and y -intercept?
20. Find an equation for the line with the given property.
- It passes through the point $(3, -6)$ and is parallel to the line $3x + y - 10 = 0$.
 - It has x -intercept 6 and y -intercept 4.
21. A geologist uses a probe to measure the temperature T (in $^{\circ}\text{C}$) of the soil at various depths below the surface, and finds that at a depth of x cm, the temperature is given by the linear equation $T = 0.08x - 4$.
- What is the temperature at a depth of one meter (100 cm)?
 - Sketch a graph of the linear equation.
 - What do the slope, the x -intercept, and T -intercept of the graph of this equation represent?
22. The maximum weight M that can be supported by a beam is jointly proportional to its width w and the square of its height h , and inversely proportional to its length L .
- Write an equation that expresses this proportionality.
 - Determine the constant of proportionality if a beam 4 in. wide, 6 in. high, and 12 ft long can support a weight of 4800 lb.
 - If a 10-ft beam made of the same material is 3 in. wide and 10 in. high, what is the maximum weight it can support?



If you had difficulty with any of these problems, you may wish to review the section of this chapter indicated below.

**If you had trouble with
this test problem**

Review this section

1	Section 1.1
2, 3, 4(a), 4(b), 4(c)	Section 1.2
4(d), 4(e), 4(f), 5	Section 1.4
6, 7	Section 1.3
8	Section 1.5
9, 10	Section 1.6
11, 12, 13	Section 1.7
14	Section 1.9
15, 16, 17(a), 17(b)	Section 1.8
17(c), 17(d)	Section 1.10
17(e), 17(f), 18	Section 1.8
19, 20, 21	Section 1.10
22	Section 1.11

Fitting Lines to Data

A model is a representation of an object or process. For example, a toy Ferrari is a model of the actual car; a road map is a model of the streets in a city. A **mathematical model** is a mathematical representation (usually an equation) of an object or process. Once a mathematical model is made it can be used to obtain useful information or make predictions about the thing being modeled. In these *Focus on Modeling* sections we explore different ways in which mathematics is used to model real-world phenomena.

▼ The Line That Best Fits the Data



In Section 1.10 we used linear equations to model relationships between varying quantities. In practice, such relationships are discovered by collecting data. But real-world data seldom fall into a precise line. The **scatter plot** in Figure 1(a) shows the result of a study on childhood obesity. The graph plots the body mass index (BMI) versus the number of hours of television watched per day for 25 adolescent subjects. Of course, we would not expect the data to be exactly linear as in Figure 1(b). But there is a linear *trend* indicated by the blue line in Figure 1(a): The more hours a subject watches TV the higher the BMI. In this section we learn how to find the line that best fits the data.

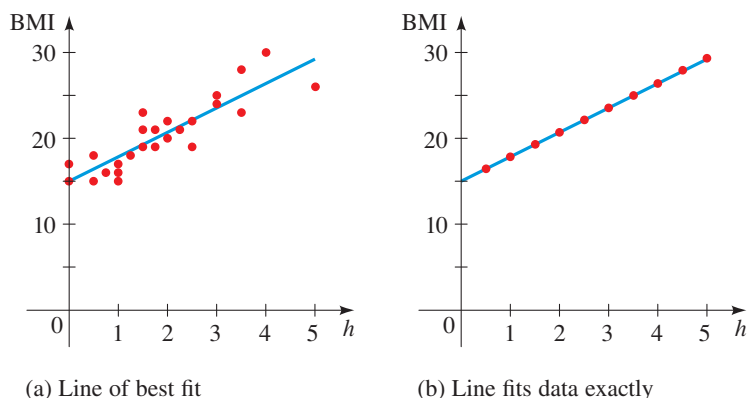


FIGURE 1

Table 1 gives the nationwide infant mortality rate for the period from 1950 to 2000. The *rate* is the number of infants who die before reaching their first birthday, out of every 1000 live births.

TABLE 1
U.S. Infant Mortality

Year	Rate
1950	29.2
1960	26.0
1970	20.0
1980	12.6
1990	9.2
2000	6.9

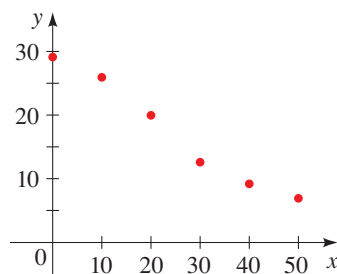


FIGURE 2 U.S. infant mortality rate

The scatter plot in Figure 2 shows that the data lie roughly on a straight line. We can try to fit a line visually to approximate the data points, but since the data aren't *exactly*

linear, there are many lines that might seem to work. Figure 3 shows two attempts at “eyeballing” a line to fit the data.

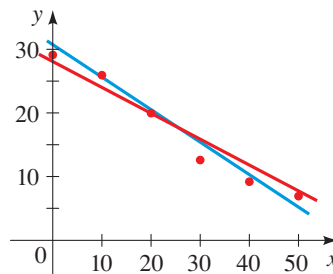


FIGURE 3 Visual attempts to fit line to data

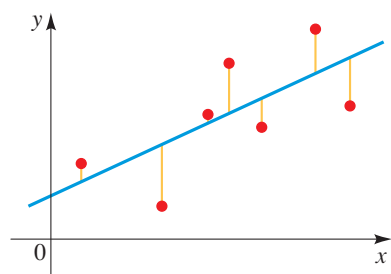


FIGURE 4 Distance from the data points to the line

Of all the lines that run through these data points, there is one that “best” fits the data, in the sense that it provides the most accurate linear model for the data. We now describe how to find this line.

It seems reasonable that the line of best fit is the line that is as close as possible to all the data points. This is the line for which the sum of the vertical distances from the data points to the line is as small as possible (see Figure 4). For technical reasons it is better to use the line where the sum of the squares of these distances is smallest. This is called the **regression line**. The formula for the regression line is found by using calculus, but fortunately, the formula is programmed into most graphing calculators. In Example 1 we see how to use a TI-83 calculator to find the regression line for the infant mortality data described above. (The process for other calculators is similar.)

EXAMPLE 1 | Regression Line for U.S. Infant Mortality Rates

- Find the regression line for the infant mortality data in Table 1.
- Graph the regression line on a scatter plot of the data.
- Use the regression line to estimate the infant mortality rates in 1995 and 2006.

SOLUTION

- To find the regression line using a TI-83 calculator, we must first enter the data into the lists L_1 and L_2 , which are accessed by pressing the **STAT** key and selecting **Edit**. Figure 5 shows the calculator screen after the data have been entered. (Note that we are letting $x = 0$ correspond to the year 1950, so that $x = 50$ corresponds to 2000. This makes the equations easier to work with.) We then press the **STAT** key again and select **Calc**, then **4:LinReg(ax+b)**, which provides the output shown in Figure 6(a). This tells us that the regression line is

$$y = -0.48x + 29.4$$

Here x represents the number of years since 1950, and y represents the corresponding infant mortality rate.

- The scatter plot and the regression line have been plotted on a graphing calculator screen in Figure 6(b).

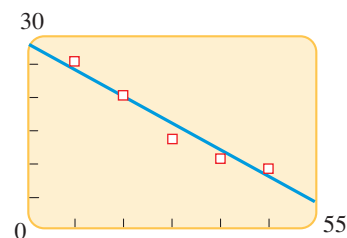
L1	L2	L3	1
0	29.2	-----	
10	26		
20	20		
30	12.6		
40	9.2		
50	6.9		

L2(7)=			

FIGURE 5 Entering the data

```
LinReg
y=ax+b
a=-.4837142857
b=29.40952381
```

(a) Output of the **LinReg** command



(b) Scatter plot and regression line

FIGURE 6

- (c) The year 1995 is 45 years after 1950, so substituting 45 for x , we find that $y = -0.48(45) + 29.4 = 7.8$. So the infant mortality rate in 1995 was about 7.8. Similarly, substituting 56 for x , we find that the infant mortality rate predicted for 2006 was about $-0.48(56) + 29.4 \approx 2.5$. ■

An Internet search shows that the actual infant mortality rate was 7.6 in 1995 and 6.4 in 2006. So the regression line is fairly accurate for 1995 (the actual rate was slightly lower than the predicted rate), but it is considerably off for 2006 (the actual rate was more than twice the predicted rate). The reason is that infant mortality in the United States stopped declining and actually started rising in 2002, for the first time in more than a century. This shows that we have to be very careful about extrapolating linear models outside the domain over which the data are spread.



Steven Hooker, 2008 Olympic gold medal winner, men's pole vault

▼ Examples of Regression Analysis

Since the modern Olympic Games began in 1896, achievements in track and field events have been improving steadily. One example in which the winning records have shown an upward linear trend is the pole vault. Pole vaulting began in the northern Netherlands as a practical activity: When traveling from village to village, people would vault across the many canals that crisscrossed the area to avoid having to go out of their way to find a bridge. Households maintained a supply of wooden poles of lengths appropriate for each member of the family. Pole vaulting for height rather than distance became a collegiate track and field event in the mid-1800s and was one of the events in the first modern Olympics. In the next example we find a linear model for the gold-medal-winning records in the men's Olympic pole vault.

EXAMPLE 2 | Regression Line for Olympic Pole Vault Records

Table 2 gives the men's Olympic pole vault records up to 2004.

- Find the regression line for the data.
- Make a scatter plot of the data, and graph the regression line. Does the regression line appear to be a suitable model for the data?
- What does the slope of the regression line represent?
- Use the model to predict the winning pole vault height for the 2008 Olympics.

TABLE 2
Men's Olympic Pole Vault Records

Year	x	Gold medalist	Height (m)	Year	x	Gold medalist	Height (m)
1896	-4	William Hoyt, USA	3.30	1956	56	Robert Richards, USA	4.56
1900	0	Irving Baxter, USA	3.30	1960	60	Don Bragg, USA	4.70
1904	4	Charles Dvorak, USA	3.50	1964	64	Fred Hansen, USA	5.10
1906	6	Fernand Gonder, France	3.50	1968	68	Bob Seagren, USA	5.40
1908	8	A. Gilbert, E. Cook, USA	3.71	1972	72	W. Nordwig, E. Germany	5.64
1912	12	Harry Babcock, USA	3.95	1976	76	Tadeusz Slusarski, Poland	5.64
1920	20	Frank Foss, USA	4.09	1980	80	W. Kozakiewicz, Poland	5.78
1924	24	Lee Barnes, USA	3.95	1984	84	Pierre Quinon, France	5.75
1928	28	Sabin Can, USA	4.20	1988	88	Sergei Bubka, USSR	5.90
1932	32	William Miller, USA	4.31	1992	92	M. Tarassob, Unified Team	5.87
1936	36	Earle Meadows, USA	4.35	1996	96	Jean Jaffione, France	5.92
1948	48	Guinn Smith, USA	4.30	2000	100	Nick Hysong, USA	5.90
1952	52	Robert Richards, USA	4.55	2004	104	Timothy Mack, USA	5.95

```
LinReg
y=ax+b
a=.0265652857
b=3.400989881
```

Output of the LinReg
function on the TI-83

SOLUTION

- (a) Let $x = \text{year} - 1900$, so 1896 corresponds to $x = -4$, 1900 to $x = 0$, and so on. Using a calculator, we find the following regression line:

$$y = 0.0266x + 3.40$$

- (b) The scatter plot and the regression line are shown in Figure 7. The regression line appears to be a good model for the data.
- (c) The slope is the average rate of increase in the pole vault record per year. So on average, the pole vault record increased by 0.0266 m/yr.

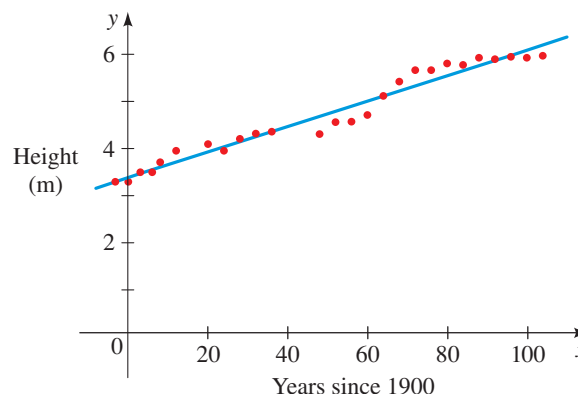


FIGURE 7 Scatter plot and regression line for pole vault data

- (d) The year 2008 corresponds to $x = 108$ in our model. The model gives

$$\begin{aligned} y &= 0.0266(108) + 3.40 \\ &\approx 6.27 \end{aligned}$$

So the model predicts that in 2008 the winning pole vault will be 6.27 m. ■

At the 2008 Olympics in Beijing, China, the men's Olympic gold medal in the pole vault was won by Steven Hooker of Australia, with a vault of 5.96 m. Although this height set an Olympic record, it was considerably lower than the 6.27 m predicted by the model of Example 2. In Problem 10 we find a regression line for the pole vault data from 1972 to 2004. Do the problem to see whether this restricted set of more recent data provides a better predictor for the 2008 record.

Is a linear model really appropriate for the data of Example 2? In subsequent *Focus on Modeling* sections, we study regression models that use other types of functions, and we learn how to choose the best model for a given set of data.

In the next example we see how linear regression is used in medical research to investigate potential causes of diseases such as cancer.

EXAMPLE 3 Regression Line for Links Between Asbestos and Cancer

When laboratory rats are exposed to asbestos fibers, some of the rats develop lung tumors. Table 3 lists the results of several experiments by different scientists.

- (a) Find the regression line for the data.
- (b) Make a scatter plot and graph the regression line. Does the regression line appear to be a suitable model for the data?
- (c) What does the y -intercept of the regression line represent?

TABLE 3
Asbestos–Tumor Data

Asbestos exposure (fibers/mL)	Percent that develop lung tumors
50	2
400	6
500	5
900	10
1100	26
1600	42
1800	37
2000	28
3000	50

**SOLUTION**

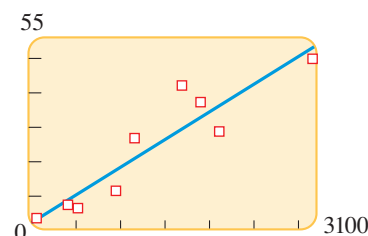
- (a) Using a calculator, we find the following regression line (see Figure 8(a)):

$$y = 0.0177x + 0.5405$$

- (b) The scatter plot and regression line are graphed in Figure 8(b). The regression line appears to be a reasonable model for the data.

```
LinReg
y=ax+b
a=.0177212141
b=.5404689256
```

(a) Output of the LinReg command



(b) Scatter plot and regression line

FIGURE 8 Linear regression for the asbestos–tumor data

- (c) The y-intercept is the percentage of rats that develop tumors when no asbestos fibers are present. In other words, this is the percentage that normally develop lung tumors (for reasons other than asbestos).

▼ How Good Is the Fit? The Correlation Coefficient

For any given set of two-variable data it is always possible to find a regression line, even if the data points do not tend to lie on a line and even if the variables don't seem to be related at all. Look at the three scatter plots in Figure 9. In the first scatter plot, the data points lie close to a line. In the second plot, there is still a linear trend but the points are more scattered. In the third plot there doesn't seem to be any trend at all, linear or otherwise.

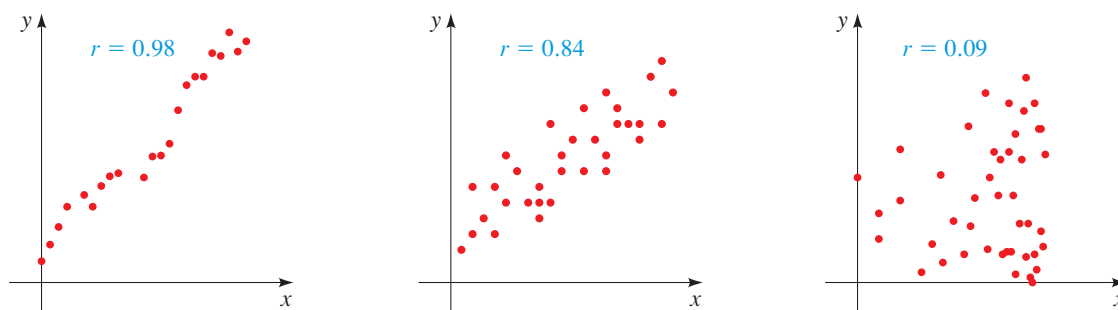


FIGURE 9

A graphing calculator can give us a regression line for each of these scatter plots. But how well do these lines represent or “fit” the data? To answer this question, statisticians have invented the **correlation coefficient**, usually denoted r . The correlation coefficient is a number between -1 and 1 that measures how closely the data follow the regression line—or, in other words, how strongly the variables are **correlated**. Many graphing calculators give the value of r when they compute a regression line. If r is close to -1

or 1, then the variables are strongly correlated—that is, the scatter plot follows the regression line closely. If r is close to 0, then the variables are weakly correlated or not correlated at all. (The sign of r depends on the slope of the regression line.) The correlation coefficients of the scatter plots in Figure 9 are indicated on the graphs. For the first plot, r is close to 1 because the data are very close to linear. The second plot also has a relatively large r , but it is not as large as the first, because the data, while fairly linear, are more diffuse. The third plot has an r close to 0, since there is virtually no linear trend in the data.

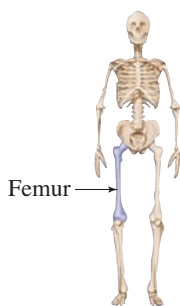
There are no hard and fast rules for deciding what values of r are sufficient for deciding that a linear correlation is “significant.” The correlation coefficient is only a rough guide in helping us decide how much faith to put into a given regression line. In Example 1 the correlation coefficient is -0.99 , indicating a very high level of correlation, so we can safely say that the drop in infant mortality rates from 1950 to 2000 was strongly linear. (The value of r is negative, since infant mortality trended *down* over this period.) In Example 3 the correlation coefficient is 0.92 , which also indicates a strong correlation between the variables. So exposure to asbestos is clearly associated with the growth of lung tumors in rats. Does this mean that asbestos *causes* lung cancer?

If two variables are correlated, it does not necessarily mean that a change in one variable *causes* a change in the other. For example, the mathematician John Allen Paulos points out that shoe size is strongly correlated to mathematics scores among schoolchildren. Does this mean that big feet cause high math scores? Certainly not—both shoe size and math skills increase independently as children get older. So it is important not to jump to conclusions: Correlation and causation are not the same thing. Correlation is a useful tool in bringing important cause-and-effect relationships to light; but to prove causation, we must explain the mechanism by which one variable affects the other. For example, the link between smoking and lung cancer was observed as a correlation long before science found the mechanism through which smoking causes lung cancer.

PROBLEMS

- 1. Femur Length and Height** Anthropologists use a linear model that relates femur length to height. The model allows an anthropologist to determine the height of an individual when only a partial skeleton (including the femur) is found. In this problem we find the model by analyzing the data on femur length and height for the eight males given in the table.

- Make a scatter plot of the data.
- Find and graph a linear function that models the data.
- An anthropologist finds a femur of length 58 cm. How tall was the person?



Femur length (cm)	Height (cm)
50.1	178.5
48.3	173.6
45.2	164.8
44.7	163.7
44.5	168.3
42.7	165.0
39.5	155.4
38.0	155.8

- 2. Demand for Soft Drinks** A convenience store manager notices that sales of soft drinks are higher on hotter days, so he assembles the data in the table.

- Make a scatter plot of the data.
- Find and graph a linear function that models the data.

- (c) Use the model to predict soft drink sales if the temperature is 95°F.

High temperature (°F)	Number of cans sold
55	340
58	335
64	410
68	460
70	450
75	610
80	735
84	780



- 3. Tree Diameter and Age** To estimate ages of trees, forest rangers use a linear model that relates tree diameter to age. The model is useful because tree diameter is much easier to measure than tree age (which requires special tools for extracting a representative cross section of the tree and counting the rings). To find the model, use the data in the table, which were collected for a certain variety of oaks.

- (a) Make a scatter plot of the data.
 (b) Find and graph a linear function that models the data.
 (c) Use the model to estimate the age of an oak whose diameter is 18 in.

Diameter (in.)	Age (years)
2.5	15
4.0	24
6.0	32
8.0	56
9.0	49
9.5	76
12.5	90
15.5	89

- 4. Carbon Dioxide Levels** The Mauna Loa Observatory, located on the island of Hawaii, has been monitoring carbon dioxide (CO_2) levels in the atmosphere since 1958. The table lists the average annual CO_2 levels measured in parts per million (ppm) from 1984 to 2006.

- (a) Make a scatter plot of the data.
 (b) Find and graph the regression line.
 (c) Use the linear model in part (b) to estimate the CO_2 level in the atmosphere in 2005. Compare your answer with the actual CO_2 level of 379.7 that was measured in 2005.

Year	CO_2 level (ppm)
1984	344.3
1986	347.0
1988	351.3
1990	354.0
1992	356.3
1994	358.9
1996	362.7
1998	366.5
2000	369.4
2002	372.0
2004	377.5
2006	380.9

Temperature (°F)	Chirping rate (chirps/min)
50	20
55	46
60	79
65	91
70	113
75	140
80	173
85	198
90	211

5. Temperature and Chirping Crickets Biologists have observed that the chirping rate of crickets of a certain species appears to be related to temperature. The table shows the chirping rates for various temperatures.

- Make a scatter plot of the data.
- Find and graph the regression line.
- Use the linear model in part (b) to estimate the chirping rate at 100°F.

6. Extent of Arctic Sea Ice The National Snow and Ice Data Center monitors the amount of ice in the Arctic year round. The table gives approximate values for the sea ice extent in millions of square kilometers from 1980 to 2006, in two-year intervals.

- Make a scatter plot of the data.
- Find and graph the regression line.
- Use the linear model in part (b) to estimate the ice extent in the year 2010.

Year	Ice extent (million km ²)	Year	Ice extent (million km ²)
1980	7.9	1994	7.1
1982	7.4	1996	7.9
1984	7.2	1998	6.6
1986	7.6	2000	6.3
1988	7.5	2002	6.0
1990	6.2	2004	6.1
1992	7.6	2006	5.7

Flow rate (%)	Mosquito positive rate (%)
0	22
10	16
40	12
60	11
90	6
100	2

7. Mosquito Prevalence The table lists the relative abundance of mosquitoes (as measured by the mosquito positive rate) versus the flow rate (measured as a percentage of maximum flow) of canal networks in Saga City, Japan.

- Make a scatter plot of the data.
- Find and graph the regression line.
- Use the linear model in part (b) to estimate the mosquito positive rate if the canal flow is 70% of maximum.

8. Noise and Intelligibility Audiologists study the intelligibility of spoken sentences under different noise levels. Intelligibility, the MRT score, is measured as the percent of a spoken sentence that the listener can decipher at a certain noise level in decibels (dB). The table shows the results of one such test.

- Make a scatter plot of the data.
- Find and graph the regression line.
- Find the correlation coefficient. Is a linear model appropriate?
- Use the linear model in part (b) to estimate the intelligibility of a sentence at a 94-dB noise level.

Noise level (dB)	MRT score (%)
80	99
84	91
88	84
92	70
96	47
100	23
104	11

9. Life Expectancy The average life expectancy in the United States has been rising steadily over the past few decades, as shown in the table.

- Make a scatter plot of the data.
- Find and graph the regression line.
- Use the linear model you found in part (b) to predict the life expectancy in the year 2006.
- Search the Internet or your campus library to find the actual 2006 average life expectancy. Compare to your answer in part (c).

Year	Life expectancy
1920	54.1
1930	59.7
1940	62.9
1950	68.2
1960	69.7
1970	70.8
1980	73.7
1990	75.4
2000	76.9

10. Olympic Pole Vault The graph in Figure 7 indicates that in recent years the winning Olympic men's pole vault height has fallen below the value predicted by the regression line in Example 2. This might have occurred because when the pole vault was a new event, there was much room for improvement in vaulters' performances, whereas now even the best training can produce only incremental advances. Let's see whether concentrating on more recent results gives a better predictor of future records.

- Use the data in Table 2 to complete the table of winning pole vault heights. (Note that we are using $x = 0$ to correspond to the year 1972, where this restricted data set begins.)
- Find the regression line for the data in part (a).
- Plot the data and the regression line on the same axes. Does the regression line seem to provide a good model for the data?
- What does the regression line predict as the winning pole vault height for the 2008 Olympics? Compare this predicted value to the actual 2008 winning height of 5.96 m, as described on page 133. Has this new regression line provided a better prediction than the line in Example 2?

Year	x	Height (m)
1972	0	5.64
1976	4	
1980	8	
1984		
1988		
1992		
1996		
2000		
2004		

11. Olympic Swimming Records The tables give the gold medal times in the men's and women's 100-m freestyle Olympic swimming event.

- (a) Find the regression lines for the men's data and the women's data.
 (b) Sketch both regression lines on the same graph. When do these lines predict that the women will overtake the men in the event? Does this conclusion seem reasonable?

MEN

Year	Gold medalist	Time (s)
1908	C. Daniels, USA	65.6
1912	D. Kahanamoku, USA	63.4
1920	D. Kahanamoku, USA	61.4
1924	J. Weissmuller, USA	59.0
1928	J. Weissmuller, USA	58.6
1932	Y. Miyazaki, Japan	58.2
1936	F. Csik, Hungary	57.6
1948	W. Ris, USA	57.3
1952	C. Scholes, USA	57.4
1956	J. Henricks, Australia	55.4
1960	J. Devitt, Australia	55.2
1964	D. Schollander, USA	53.4
1968	M. Wenden, Australia	52.2
1972	M. Spitz, USA	51.22
1976	J. Montgomery, USA	49.99
1980	J. Woithe, E. Germany	50.40
1984	R. Gaines, USA	49.80
1988	M. Biondi, USA	48.63
1992	A. Popov, Russia	49.02
1996	A. Popov, Russia	48.74
2000	P. van den Hoogenband, Netherlands	48.30
2004	P. van den Hoogenband, Netherlands	48.17
2008	A. Bernard, France	47.21

WOMEN

Year	Gold medalist	Time (s)
1912	F. Durack, Australia	82.2
1920	E. Bleibtrey, USA	73.6
1924	E. Lackie, USA	72.4
1928	A. Osipowich, USA	71.0
1932	H. Madison, USA	66.8
1936	H. Mastenbroek, Holland	65.9
1948	G. Andersen, Denmark	66.3
1952	K. Szoke, Hungary	66.8
1956	D. Fraser, Australia	62.0
1960	D. Fraser, Australia	61.2
1964	D. Fraser, Australia	59.5
1968	J. Henne, USA	60.0
1972	S. Nielson, USA	58.59
1976	K. Ender, E. Germany	55.65
1980	B. Krause, E. Germany	54.79
1984	(Tie) C. Steinseifer, USA N. Hogshead, USA	55.92
1988	K. Otto, E. Germany	54.93
1992	Z. Yong, China	54.64
1996	L. Jingyi, China	54.50
2000	I. DeBruijn, Netherlands	53.83
2004	J. Henry, Australia	53.84
2008	B. Steffen, Germany	53.12

12. Shoe Size and Height Do you think that shoe size and height are correlated? Find out by surveying the shoe sizes and heights of people in your class. (Of course, the data for men and women should be separate.) Find the correlation coefficient.

13. Demand for Candy Bars In this problem you will determine a linear demand equation that describes the demand for candy bars in your class. Survey your classmates to determine what price they would be willing to pay for a candy bar. Your survey form might look like the sample to the left.

Would you buy a candy bar from the vending machine in the hallway if the price is as indicated?

Price	Yes or No
30¢	
40¢	
50¢	
60¢	
70¢	
80¢	
90¢	
\$1.00	
\$1.10	
\$1.20	

- (a) Make a table of the number of respondents who answered "yes" at each price level.
 (b) Make a scatter plot of your data.
 (c) Find and graph the regression line $y = mp + b$, which gives the number of respondents y who would buy a candy bar if the price were p cents. This is the *demand equation*. Why is the slope m negative?
 (d) What is the p -intercept of the demand equation? What does this intercept tell you about pricing candy bars?